

ON EMBEDDINGS OF $\text{CAT}(0)$ CUBE COMPLEXES INTO PRODUCTS OF TREES VIA COLOURING THEIR HYPERPLANES

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Abstract. We prove that the contact graph of a 2-dimensional $\text{CAT}(0)$ cube complex \mathbf{X} of maximum degree Δ can be coloured with at most $\epsilon(\Delta) = M\Delta^{15}$ colours, for a fixed constant M . This implies that \mathbf{X} (and the associated median graph) isometrically embeds in the Cartesian product of at most $\epsilon(\Delta)$ trees, and that the event structure whose domain is \mathbf{X} admits a nice labeling with $\epsilon(\Delta)$ labels. On the other hand, we present an example of a 5-dimensional $\text{CAT}(0)$ cube complex with uniformly bounded degrees of 0-cubes which cannot be embedded into a Cartesian product of a finite number of trees. This answers in the negative a question raised independently by M. Sageev and the first author of this paper.

1. INTRODUCTION

In his seminal paper [Gro87], among many other results, Gromov gave a nice combinatorial characterization of $\text{CAT}(0)$ cube complexes as simply connected cube complexes in which the links of 0-cubes are simplicial flag complexes. Subsequently, Sageev [Sag95] introduced and investigated the concept of (combinatorial) hyperplanes of $\text{CAT}(0)$ cube complexes, showing in particular that each hyperplane is itself a $\text{CAT}(0)$ cube complex and partitions the complex into two $\text{CAT}(0)$ cube complexes.

These two results identify $\text{CAT}(0)$ cube complexes as the basic objects in a “high-dimensional Bass-Serre theory”, and $\text{CAT}(0)$ and nonpositively-curved cube complexes have thus been studied extensively in geometric group theory. For instance, many well-known classes of groups are known to act nicely on $\text{CAT}(0)$ cube complexes (see, e.g. [CD95a, CD95b, Far05, NR03, Wis04]). Groups acting essentially on $\text{CAT}(0)$ cube complexes enjoy a wide variety of properties resulting from such actions – they do not have

Kazhdan’s property (T) [NR98] and many of them admit splittings or virtual splittings related to the hyperplanes (e.g. [Sag97, Nib04]), for example. CAT(0) cube complexes whose hyperplanes are related to group splittings also lie at the heart of Wise’s recent work on groups with a quasiconvex hierarchy and its attendant applications to the topology of 3-manifolds [Wis].

On the other hand, [Che00, Rol98] established that the 1-skeletons of CAT(0) cube complexes are exactly the median graphs, i.e., the graphs in which any triplet of vertices admit a unique median vertex. Median graphs and related median structures have been investigated in several contexts by quite a number of authors for more than half a century. They have many nice properties and admit numerous characterizations relating them to other discrete structures. Avann [Ava61] showed that median graphs and discrete median algebras (i.e., ternary algebras which are subdirect products of the two-element algebra $\{0, 1\}$ [BH83, Isb80]) constitute the same objects. Bandelt [Ban84] proved that median graphs are exactly the retracts of the hypercubes. Barthélemy and Constantin [BC93] showed that pointed median graphs are exactly the domains of event structures with binary conflict (investigated in computer science in concurrency theory [NPW85, WN95, RT91]), while Schaefer [Sch78] proved that median-stable subsets of Boolean algebras are exactly the solution sets of instances of the 2-SAT problem (a well-known problem in theoretical computer science). Mulder and Schrijver [MS79] characterized the split systems (bipartitions) arising from halfspaces of median graphs as the Helly copair hypergraphs, thus extending the bijection of Buneman between trees and pairwise laminar (compatible) split systems. Due to this bijection for trees, Dress, Huber, and Moulton [DHHM97] called *Buneman complexes* the median closures of arbitrary collections of splits of a finite set. For other results and characterizations, see the books [Fed95, IK00, Mul80, vdV93] and the surveys [BC08, BH83].

All CAT(0) cube complexes \mathbf{X} and median graphs – the 1-skeleta $G(\mathbf{X})$ of \mathbf{X} – are intimately related to hypercubes: they are constituted of cubes and themselves embed isometrically into hypercubes. The minimum dimension of a hypercube into which $G(\mathbf{X})$ (or \mathbf{X}) isometrically embeds equals the number of hyperplanes of \mathbf{X} , or, equivalently, the number of equivalence classes of the transitive closure of the “opposite” relation of edges of $G(\mathbf{X})$ on 2-cubes of \mathbf{X} , or, equivalently, the number of convex splits of $G(\mathbf{X})$. While the dimension of the smallest hypercube into which the median graph $G(\mathbf{X})$ embeds is easy to determine, the problem of determining the least number $\tau(\mathbf{X}) = \tau(G(\mathbf{X}))$ of tree factors necessary for an isometric embedding of the 1-skeleton of \mathbf{X} into a Cartesian product of trees is hard.

There is a canonical construction of median graphs and CAT(0) cube complexes beginning from arbitrary graphs G : namely, for a graph G the *simplex graph* $\kappa(G)$ has the simplices (the complete subgraphs) of G as its vertices and pairs of (comparable) simplices differing in exactly one vertex as its edges. The graph $\kappa(G)$ is median, moreover it was shown in [BvdV89] that $\kappa(G)$ can be isometrically embedded into the Cartesian product of at most k trees if and only if the chromatic number $\chi(G)$ of G is at most k . In particular, it is NP-complete, even for $k = 3$, to decide whether $\tau(\mathbf{X}) \leq k$ for a 2-dimensional CAT(0) cube complex (i.e. if a 3-cube-free cube complex embeds into the Cartesian product of k trees)

[BvdV89]. Departing from triangle-free Mycielski graphs G (i.e., graphs with arbitrarily high chromatic numbers), one gets, via the simplex-graph construction, 3-cube-free median graphs $\kappa(G)$ with arbitrarily large $\tau(\kappa(G))$.

For arbitrary CAT(0) cube complexes \mathbf{X} , the value $\tau(\mathbf{X})$ is closely related to the chromatic number of the so-called *incompatibility* or *crossing graph* $\Gamma_{\#}(\mathbf{X})$ of \mathbf{X} . $\Gamma_{\#}(\mathbf{X})$ can be viewed as the intersection graph of the hyperplanes of \mathbf{X} : its vertices are the hyperplanes of \mathbf{X} sensu [Sag95] and two hyperplanes are adjacent in $\Gamma_{\#}(\mathbf{X})$ if and only if they cross (or, equivalently, they intersect). The crossing graph of the CAT(0) cube complex derived from the simplex graph $\kappa(G)$ of G coincides with G (see, e.g. [Hag11, Rol98]).

Extending the fact that $\tau(\kappa(G)) = \chi(G)$, it was formally stated in [BCE10b] (and seems to be known to other people working in the field) that the equality $\tau(\mathbf{X}) = \chi(\Gamma_{\#}(\mathbf{X}))$ holds for all CAT(0) cube complexes \mathbf{X} . Since an arbitrary simplicial graph can be realized as the crossing graph of a CAT(0) cube complex \mathbf{X} , in order to better capture the structure of \mathbf{X} and some graph-parameters of its 1-skeleton $G(\mathbf{X})$, the second author of this paper introduced in [Hag11] the concept of the *contact graph* $\Gamma(\mathbf{X})$ of \mathbf{X} : the vertices of $\Gamma(\mathbf{X})$ are the hyperplanes of \mathbf{X} and two hyperplanes are adjacent in $\Gamma(\mathbf{X})$ if and only if they cross or osculate (i.e., their carriers touch each other). $\Gamma(\mathbf{X})$ can be also viewed as the intersection graph of the carriers of the hyperplanes of \mathbf{X} . The clique number $\omega(\Gamma(\mathbf{X}))$ of the contact graph of \mathbf{X} is exactly the maximum degree in $G(\mathbf{X})$ of a 0-cube of \mathbf{X} , i.e., to the maximum number of 1-cubes incident to a 0-cube of \mathbf{X} . The contact graph $\Gamma(\mathbf{X})$ always contains the crossing graph $\Gamma_{\#}(\mathbf{X})$ as a spanning subgraph. $\Gamma(\mathbf{X})$ also hosts the *pointed contact graph* $\Gamma_{\alpha}(\mathbf{X})$ of the 1-skeleton $G_{\alpha}(\mathbf{X})$ of \mathbf{X} pointed at arbitrary vertex α . The graph $\Gamma_{\alpha}(\mathbf{X})$ has hyperplanes of \mathbf{X} as vertices and two hyperplanes H, H' are adjacent in $\Gamma_{\alpha}(\mathbf{X})$ if and only if they are adjacent in $\Gamma(\mathbf{X})$ and two incident 1-cubes, one crossed by H and another crossed by H' , are directed away from the common origin.

M. Sageev and, independently, the first author of this paper asked the following question:

Question 1. *Is it true that all CAT(0) cube complexes \mathbf{X} with uniformly bounded degrees can be isometrically embedded into a finite number of trees, or, equivalently, if there exists a function $\epsilon : \mathbb{N} \mapsto \mathbb{N}$ such that $\tau(\mathbf{X}) \leq \epsilon(\Delta)$ for any CAT(0) cube complex \mathbf{X} of degree Δ ?*

This question is closely related with the conjecture of Rozoy and Thiagarajan [RT91] (also called the *nice labeling problem*) asserting that:

Question 2. *Any event structure with finite (out)degree admits a labeling with a finite number of labels.*

As noted above, pointed median graphs are exactly the domains of event structures with binary conflict [BC93]. Then, in view of the bijection between median graphs and 1-skeletons of CAT(0) cube complexes, the nice labeling problem for such event structures can be equivalently viewed as the colouring problem of the pointed contact graph $\Gamma_{\alpha}(\mathbf{X})$ of the CAT(0) cube complex \mathbf{X} associated to the domain of the event structure. Since $\chi(\Gamma_{\alpha}(\mathbf{X})) \leq \chi(\Gamma(\mathbf{X}))$ and $\chi(\Gamma_{\#}(\mathbf{X})) \leq \chi(\Gamma(\mathbf{X}))$, in relation with Questions 1 and 2, the following question is natural:

Question 3. *Is it true that the chromatic number $\chi(\Gamma(\mathbf{X}))$ of the contact graph of a $\text{CAT}(0)$ cube complex \mathbf{X} of degree Δ can be bounded by a function ϵ of Δ ?*

Since $\omega(\Gamma(\mathbf{X})) = \Delta$ and $\Gamma_{\#}(\mathbf{X}), \Gamma_{\alpha}(\mathbf{X})$ are subgraphs of $\Gamma(\mathbf{X})$, all three questions can be reformulated, namely: which of the classes of graphs $\Gamma_{\#}(\mathbf{X}), \Gamma_{\alpha}(\mathbf{X})$, and $\Gamma(\mathbf{X})$ are χ -bounded? A class \mathcal{C} of graphs is called χ -bounded if there exists a function f such that $\chi(G) \leq f(\omega(G))$ for any graph G of \mathcal{C} . For example, Asplund and Grünbaum [AG60] proved that the intersection graphs of axis-parallel rectangles of \mathbb{R}^2 are χ -bounded (we will review below some other classes of χ -bounded intersection graphs).

Via a series of nontrivial examples, Burling [Bur65] showed that the class of intersection graphs of axis-parallel boxes of \mathbb{R}^3 is not χ -bounded. Based on Burling's examples, it was recently shown in [Che11] that for $\text{CAT}(0)$ cube complexes the classes of graphs $\Gamma(\mathbf{X})$ and $\Gamma_{\alpha}(\mathbf{X})$ are not χ -bounded, thus disproving the nice labeling conjecture of [RT91]. In this paper, we adapt this counterexample by using the recubulation technique from [Hag11] to show that the class of crossing graphs $\Gamma_{\#}(\mathbf{X})$ of $\text{CAT}(0)$ cube complexes is also not χ -bounded, thus answering in the negative the first open question.

On the other hand, and this is the main contribution of our paper, we show that in the case of 2-dimensional $\text{CAT}(0)$ cube complexes \mathbf{X} the contact graphs $\Gamma(\mathbf{X})$ (and therefore the crossing and the pointed contact graphs) are χ -bounded by a polynomial function in $\omega(\Gamma(\mathbf{X})) = \Delta$, thus showing that in the 2-dimensional case the three questions have positive answers; this is the content of our main result:

Theorem 1. *Let \mathbf{X} be a 2-dimensional $\text{CAT}(0)$ cube complex such that the degrees of all its vertices are uniformly bounded by Δ . Then there exists $M < \infty$, independent of \mathbf{X} , such that $\chi(\Gamma(\mathbf{X})) \leq \epsilon(\Delta) = M\Delta^{15}$. In particular, $\tau(\mathbf{X}) \leq \epsilon(\Delta)$, i.e. the 1-skeleton of \mathbf{X} isometrically embeds into the Cartesian product of at most $\epsilon(\Delta)$ trees. Finally, all event structures of (out)degree Δ_0 whose domains are 2-dimensional cube complexes admit a nice labeling with at most $\epsilon(\Delta_0 + 2)$ labels.*

We actually obtain the following bound: $\chi(\Gamma(\mathbf{X})) \leq \epsilon(\Delta) = \frac{140417}{32}\Delta^{15}$, or, simply $M < 4389$.

The second assertion of Theorem 1 follows from the first assertion because $\Gamma_{\#}(\mathbf{X})$ is a subgraph of $\Gamma(\mathbf{X})$ and because of the equality $\tau(\mathbf{X}) = \chi(\Gamma_{\#}(\mathbf{X}))$. The third assertion is a consequence of the fact that the number of labels in a nice labeling is equal to $\chi(\Gamma_{\alpha}(\mathbf{X}))$, and because $\Gamma_{\alpha}(\mathbf{X})$ is a subgraph of $\Gamma(\mathbf{X})$ and $\Delta \leq \Delta_0 + n$ holds for all n -dimensional $\text{CAT}(0)$ cube complexes.

The main focus of our paper is thus to prove the first assertion of Theorem 1. To show that the chromatic number $\chi(\Gamma(\mathbf{X}))$ of the contact graph $\Gamma(\mathbf{X})$ is polynomially bounded in Δ , we show that the edges of $\Gamma(\mathbf{X})$ can be distributed over six spanning subgraphs of $\Gamma(\mathbf{X})$, such that the chromatic numbers of each of these subgraphs can be polynomially bounded. As a result, each (vertex) hyperplane of $\Gamma(\mathbf{X})$ receives a sextuple of colours, each colour corresponding to the colour received by this vertex in the colouring of the corresponding subgraph. Since each edge of $\Gamma(\mathbf{X})$ is present in at least one spanning subgraph, the sextuple-colouring of the

hyperplanes of \mathbf{X} is a correct colouring of the contact graph $\Gamma(\mathbf{X})$. The number of colours is the product of the six numbers of colours used to colour the spanning subgraphs, whence it is polynomial in Δ . In Sections 4-6, one after another, we will define and colour the six spanning subgraphs. For this, we will study the geometrical and the combinatorial properties of contact graphs of 2-dimensional CAT(0) cube complexes.

We conclude with the formulation of the second principal result:

Theorem 2. *For any $n > 0$, there exists a finite CAT(0) cube complex \mathbf{X}_n with constant maximum degree such that any colouring of the crossing graph of \mathbf{X}_n requires more than n colours, i.e., any isometric embedding of \mathbf{X}_n into a Cartesian product of trees requires $> n$ trees. There exists an infinite CAT(0) cube complex \mathbf{X} with constant maximum degree which cannot be isometrically embedded into a Cartesian product of a finite number of trees, i.e., the chromatic number of its crossing graph is infinite.*

2. RELATED RESULTS

Our counterexample in Theorem 2 and some steps of the proof of Theorem 1 are based on the fact that there exist classes of geometric intersection graphs that are χ -bounded, and also classes that are not χ -bounded. Therefore, we continue with a brief review of such classes. Given a family of sets \mathcal{F} with the ground-set S , the *intersection graph* of \mathcal{F} has the sets of \mathcal{F} as the vertex-set and two sets F, F' define an edge of the intersection graph if and only if $F \cap F' \neq \emptyset$. With some abuse of notation, we will denote by $\chi(\mathcal{F})$ and $\omega(\mathcal{F})$, respectively, the *chromatic number* and the *clique number* of the intersection graph of \mathcal{F} . The *degree* $\delta(\mathcal{F})$ of \mathcal{F} is the maximum number of sets of \mathcal{F} to which an element of S belongs. It is evident and well-known that the equality $\delta(\mathcal{F}) = \omega(\mathcal{F})$ holds if \mathcal{F} satisfies the *Helly property*, i.e., any subfamily \mathcal{F}' of \mathcal{F} has a nonempty intersection whenever any two sets of \mathcal{F}' intersect.

Gallai established (in unpublished work; see [Gol80, GLB03]) that $\chi(\mathcal{I}) = \omega(\mathcal{I})$ for families \mathcal{I} of intervals of the real line (whose intersection graphs are called interval graphs). This founding result has numerous generalizations, among which we recall only a few of them. First, it is well known that the equality

$$\chi(\mathcal{T}) = \omega(\mathcal{T}) = \delta(\mathcal{T})$$

holds for families \mathcal{T} of subtrees of a tree [Gol80, GLB03] (the intersection graphs of subtrees are the so-called *chordal graphs* which are known to be a subclass of perfect graphs).

On the other hand, Asplund and Grunbaum [AG60] showed that $\chi(\mathcal{R}) \leq 4\omega^2(\mathcal{R}) - 4\omega(\mathcal{R})$ for any family \mathcal{R} of axis-parallel rectangles of \mathbb{R}^2 . Burling [Bur65] presented a series \mathcal{B}_n of axis-parallel boxes of \mathbb{R}^3 with $\omega(\mathcal{B}_n) = 2$ and $\chi(\mathcal{B}_n) > n$ (see [GLB03] for a description of Burling's construction). Gyárfás [Gyá85] showed that $\chi(\mathcal{I}_t) \leq 2t\omega(\mathcal{I}_t)$ for families of sets each set consisting of t intervals of the line. Gyárfás [Gyá85] and Kostochka [Kos88] showed that the class of intersection graphs of chords of a circle is χ -bounded by $2^\omega\omega(\omega + 2)$; there are known examples showing only that $\chi \geq \omega \log \omega$ (similar kinds of bounds were proved in [KK97] for polygon-circle graphs). On the other hand, Kostochka [Kos88] proved that the chromatic number of any triangle-free intersection graph of chords is at most 5 (and this

bound is known to be sharp). It is conjectured in [GL85] that the class of intersection graphs of “L” shapes in the plane is χ -bounded and McGuinness [McG96] established this conjecture in the case of L-shapes whose vertical stem is infinite.

We conclude with a few known results about the three questions in case of CAT(0) cube complexes. Using the result of Kostochka about the triangle-free graphs of chords and the “stretchability” of hyperplanes of plane 2-dimensional CAT(0) cube complexes (called square-graphs), it was shown in [BCE10a] that 1-skeletons of such graphs can be embedded into Cartesian products of at most 5 trees. In [BCE10b] the CAT(0) cube complexes which can be embedded in Cartesian products of two trees were characterized in a local-to-global way as the 2-dimensional CAT(0) cube complexes in which the links of vertices are bipartite graphs. In unpublished work, Sageev has shown that Gromov-hyperbolic CAT(0) cube complexes (and in particular their 1-skeletons) isometrically embed in the product of finitely many trees, and Sageev and Druţu have extended this to certain CAT(0) cube complexes that are universal covers of nonpositively-curved cube complexes with relatively hyperbolic fundamental group (personal communication from M. Sageev).

Likewise, the 1-skeleton of an *acyclic* CAT(0) cube complex of dimension d admits an isometric embedding in the product of at most d trees [BC96]. The same paper also introduces the notion of a *perfect* CAT(0) cube complex as a CAT(0) cube complex \mathbf{X} whose crossing graph $\Gamma_{\#}(\mathbf{X})$ is perfect (i.e., the chromatic number of $\Gamma_{\#}(\mathbf{X})$ and of any of its induced subgraphs equals the clique number) and conjectures that a CAT(0) cube complex \mathbf{X} is perfect if and only if the CAT(0) cube complexes which correspond to simplex graphs obtained via median homomorphisms from $G(\mathbf{X})$ are perfect and shows that the *Strong Perfect Graph Conjecture* implies this conjecture. The Strong Perfect Graph Conjecture has since been proved [CRST06], and thus the conjectured characterization of perfect CAT(0) cube complexes is also true.

Relatedly, Ballman and Świątkowski, in [BS99], showed that CAT(0) cube complexes with some additional structure – *foldable cubical chamber complexes* – admit bi-Lipschitz embeddings into the product of d trees, where d is the dimension of the cube complex.

On the other hand, there are known to be several classes of event structures for which the nice labeling conjecture is true. Assous et al. [ABCR94] proved that the event structures of degree 2 admit nice labelings with 2 labels and noticed that Dilworth’s theorem implies that the conflict-free event structures of degree n have nice labelings with n labels. Recently, Santocanale [San10] proved that all event structures of degree 3 and with tree-like partial orders have nice labelings with 3 labels.

3. PRELIMINARIES

This section is devoted to definitions and basic properties of the objects used throughout the paper. We begin with a brief review of graph colouring, and then discuss the basic properties of CAT(0) cube complexes (following the discussion in [Hag11]) and median graphs (following the discussion in [BC08]). We then define the crossing graph (see e.g. [Rol98, Nib04]) and the contact graph (see [Hag11]) of a CAT(0) cube complex, and the pointed

contact graph (see [Che11]) of a pointed CAT(0) complex or a pointed median graph. This is then related to the nice labeling problem for event structures. We discuss disc diagrams in CAT(0) cube complexes, which are used throughout the paper, and then relate the crossing and contact graphs to isometric and convex embeddings of CAT(0) cube complexes. Finally, we define and prove basic properties of hyperplane-distance, footprints, and imprints, all of which are used in our colouring of contact graphs.

3.1. Graph colouring. Let G be a connected graph, with vertex set $\mathcal{V}(G)$. An edge of G joining $x, y \in \mathcal{V}(G)$ is denoted xy . A *graph homomorphism* $\phi : G \rightarrow H$ is a map from $\mathcal{V}(G)$ to $\mathcal{V}(H)$ such that, if xy is an edge of G , then $\phi(x)\phi(y)$ is an edge of H . A *colouring* of G by a set \mathcal{K} of colours is a graph homomorphism $c : G \rightarrow K(\mathcal{K})$, where $K(\mathcal{K}) = K_n$ is the complete graph with vertex set \mathcal{K} of cardinality n . Equivalently, c is an assignment of an element of \mathcal{K} – a colour – to each vertex of G in such a way that, for each edge xy , we have $c(x) \neq c(y)$. The *chromatic number* $\chi(G)$ of G is the cardinality of a smallest set \mathcal{K} for which there exists a \mathcal{K} -colouring of G . Note that if $H \subseteq G$ is a subgraph, then $\chi(H) \leq \chi(G)$. Also, it was shown by de Bruijn and Erdős [dBE51] that, for any graph G , we have $\chi(G) \leq n$ if and only if $\chi(H) \leq n$ for each finite subgraph H of G . Hence, to \mathcal{K} -colour G , it suffices to fix an arbitrary vertex v and to \mathcal{K} -colour the ball $B_n(v)$ for each $n \geq 0$.

3.2. CAT(0) cube complexes and hyperplanes. For $0 \leq d < \infty$, a d -cube is a copy of $[-\frac{1}{2}, \frac{1}{2}]$ endowed with the ℓ_1 metric. A *cube complex* is a CW-complex \mathbf{X} whose cells are cubes of various dimensions, attached in the expected way: any two cubes of \mathbf{X} that have nonempty intersection intersect in a common face, i.e. the attaching map of each cube restricts to a combinatorial isometry on its faces.

The *link* of a 0-cube $x \in \mathbf{X}$ is the complex built of simplices, with a $(d-1)$ -simplex for each d -cube containing x , with simplices attached according to the attachments of the corresponding cubes. The simply-connected cube complex \mathbf{X} is *CAT(0)* if the link $Lk(x)$ of each 0-cube x is a *flag (simplicial) complex*, i.e. if each $(d+1)$ -clique in $Lk(x)$ spans an d -simplex. The *dimension* of the CAT(0) cube complex \mathbf{X} is the largest value of d for which \mathbf{X} contains a d -cube, and the *degree* Δ of \mathbf{X} is the degree of a highest-degree 0-cube.

A *midcube* of the d -cube c , with $d \geq 1$, is the isometric subspace obtained by restricting exactly one of the coordinates of d to 0. Note that a midcube is a $(d-1)$ -cube. The midcubes a and b of \mathbf{X} are *adjacent* if they have a common face, and a *hyperplane* H of \mathbf{X} is a subspace that is a maximal connected union of midcubes such that, if $a, b \subset H$ are midcubes, either a and b are disjoint or they are adjacent. Equivalently, a hyperplane H is a maximal connected union of midcubes such that, for each cube c , either $H \cap c = \emptyset$ or $H \cap c$ is a single midcube of c . In [Sag95], Sageev showed that each hyperplane H is a CAT(0) cube complex of dimension at most $\dim \mathbf{X} - 1$, and that $\mathbf{X} - H$ consists of exactly two components, called *halfspaces*. A 1-cube c is *dual* to the hyperplane H if the 0-cubes of c lie in distinct halfspaces of H , i.e. if the midpoint of c is a midcube contained in H . The relation “dual to the same hyperplane” is an equivalence relation on the set of 1-cubes of \mathbf{X} ; denote this relation by Θ and denote by $\Theta(H)$ the equivalence class consisting of 1-cubes dual to the hyperplane H .

In the remainder of this paper, \mathbf{X} denotes a CAT(0) cube complex and \mathcal{H} denotes the set of hyperplanes. For each $H \in \mathcal{H}$, let $N(H)$ be the subcomplex of \mathbf{X} consisting of all closed d -cubes c such that $H \cap c \neq \emptyset$. The subcomplex $N(H)$ is called the *carrier* of H , and it was proved in [Sag95] that $N(H)$ is a CAT(0) cube complex isomorphic to $[-\frac{1}{2}, \frac{1}{2}] \times H$. In particular, we shall often use the natural projection $N(H) \rightarrow H \cong \{0\} \times H$ arising from the isomorphism $N(H) \cong [-\frac{1}{2}, \frac{1}{2}] \times H$.

Although \mathbf{X} admits a CAT(0) metric arising from the ℓ_2 metric on the constituent cubes [Gro87], it is more natural to use the *wall-metric* arising from the hyperplanes, discussed, for example, in [Hag11] and, in the context of median complexes, in [vdV93]. More precisely, \mathbf{X} admits a metric d such that the restriction of d to any cube c of \mathbf{X} is the ℓ_1 metric on c and the restriction of d to the 1-skeleton of \mathbf{X} is the standard graph distance. In particular, a *combinatorial path* $P \rightarrow \mathbf{X}$ – a path in the 1-skeleton of \mathbf{X} – is a geodesic segment in \mathbf{X} if and only if P is a geodesic segment of the 1-skeleton. Equivalently, P is a geodesic segment if and only if P contains at most one 1-cube dual to each hyperplane of \mathbf{X} . The length $|P|$ of the path P is equal to the number of hyperplanes (with multiplicity) that cross P in the sense defined below, and P is a geodesic segment if and only if $|P|$ is equal to the number of hyperplanes that separate its endpoints (as defined below). This ℓ^1 metric was investigated by Haglund in [Hag07].

Let $Y \subseteq \mathbf{X}$ be a subcomplex. Let H be a hyperplane, and denote by $\mathbf{A}(H)$ and $\mathbf{B}(H)$ the halfspaces of H . Then H *crosses* Y if $\mathbf{A}(H) \cap Y$ and $\mathbf{B}(H) \cap Y$ are both nonempty. In particular, if H' is another hyperplane, then H and H' *cross* if and only if each of the *quarter-spaces* $\mathbf{A}(H) \cap \mathbf{A}(H')$, $\mathbf{A}(H) \cap \mathbf{B}(H')$, $\mathbf{B}(H) \cap \mathbf{A}(H')$, $\mathbf{B}(H) \cap \mathbf{B}(H')$ is nonempty. Equivalently, H and H' cross if and only if there exists a 2-cube s whose boundary path contains a concatenation cc' , where $c \in \Theta(H)$ and $c' \in \Theta(H')$.

If H and H' do not cross, but \mathbf{X} contains a pair of 1-cubes $c \in \Theta(H)$, $c' \in \Theta(H')$ such that c, c' have a common 0-cube, then H and H' *osculate*. If H and H' either cross or osculate, then they *contact*, denoted $H \lrcorner H'$. Equivalently, $H \lrcorner H'$ if and only if $N(H) \cap N(H') \neq \emptyset$.

If Y, Y' are convex subcomplexes of \mathbf{X} , and H is a hyperplane such that $Y \subseteq \mathbf{A}(H)$ and $Y' \subseteq \mathbf{B}(H)$, then H *separates* Y from Y' . We see that $H \lrcorner H'$ if and only if no third hyperplane separates H from H' . Relatedly, we say that a subset $\mathcal{H}' \subseteq \mathcal{H}$ is *inseparable* if, given any two hyperplanes $H, H' \in \mathcal{H}'$, every hyperplane $H'' \in \mathcal{H}$ that separates H from H' also belongs to \mathcal{H}' . Also, the distance between the convex subcomplexes Y and Y' is equal to the number of hyperplanes that separate Y from Y' , and this is also the length of a shortest geodesic segment having one endpoint in Y and one endpoint in Y' .

Each hyperplane H , and its carrier $N(H)$, is convex with respect to the wall-metric, and we give a simple characterization of convexity below. Sageev [Sag95] also showed that H is convex with respect to the CAT(0) metric.

The property of being a *Cartesian product* is characterized as follows for CAT(0) cube complexes. The Cartesian product $\mathbf{X} = \mathbf{X}_1 \times \mathbf{X}_2$ is again a CAT(0) cube complex. Let \mathcal{H}_1 and \mathcal{H}_2 denote the sets of hyperplanes of \mathbf{X}_1 and \mathbf{X}_2 respectively. Then each hyperplane of \mathbf{X} is of the form $H \times \mathbf{X}_2$, with $H \in \mathcal{H}_1$, or $\mathbf{X}_1 \times H$, where $H \in \mathcal{H}_2$, and each hyperplane

of the former form crosses each hyperplane of the latter form. Conversely, if the set \mathcal{H} of hyperplanes of \mathbf{X} decomposes as a disjoint union $\mathcal{H} = \mathcal{H}_1 \sqcup \mathcal{H}_2$ such that each element of \mathcal{H}_1 crosses each element of \mathcal{H}_2 , then $\mathbf{X} \cong \mathbf{X}_1 \times \mathbf{X}_2$, where for $i \in \{1, 2\}$, the complex \mathbf{X}_i is a convex subcomplex of \mathbf{X} that is crossed by each hyperplane in \mathcal{H}_i and no others. The wall-metric on \mathbf{X} is identical to the metric defined by $d((x_1, x_2), (y_1, y_2)) = d_1(x_1, y_1) + d_2(x_2, y_2)$, where d_i is the wall-metric on \mathbf{X}_i and $x_i, y_i \in \mathbf{X}_i$.

A major source of CAT(0) cube complexes comes from the notion of a “cube complex dual to a wallspace”. A *wallspace* is a set \mathcal{S} , together with a collection \mathcal{H} of bipartitions of \mathcal{S} , called *walls*, into *halfspaces*. We require that for all $s_1, s_2 \in \mathcal{S}$, there are finitely many walls H such that s_1 and s_2 lie in different halfspaces associated to H .

The dual cube complex \mathbf{X} is constructed as follows: a 0-cube v is a choice $v(H)$ of halfspace for each $H \in \mathcal{H}$, in such a way that $v(H) \cap v(H') \neq \emptyset$ for all $H, H' \in \mathcal{H}$. Each $s \in \mathcal{S}$ determines a *canonical* 0-cube v_s defined by declaring, for each $H \in \mathcal{H}$, that $v_s(H)$ is the halfspace containing s . The set of 0-cubes of the dual cube complex consists of the set of canonical 0-cubes, together with any 0-cube v that differs from some, and hence any, canonical 0-cube on finitely many walls.

\mathbf{X} has a set of hyperplanes corresponding bijectively to \mathcal{H} . The construction of a dual cube complex from a wallspace appears in a group-theoretic context in the work of Sageev [Sag95]; the formal notion of a wallspace is due to Haglund and Paulin [HP98], and the construction of the cube complex in this more general context appears in [CN05] and [Nic04].

This construction shows that each 0-cube of \mathbf{X} can be thought of as a consistent choice of halfspace for each hyperplane, which is to say that, if v is a 0-cube of \mathbf{X} and H, H' are hyperplanes, then $v(H) \cap v(H') \neq \emptyset$, where $v(H)$ denotes the halfspace of H containing v . Moreover, since \mathbf{X} is connected, any two 0-cubes are separated by finitely many hyperplanes, so that, for 0-cubes v, v' , there are finitely many hyperplanes H such that $v(H) \neq v'(H)$.

Hence the *pointed CAT(0) cube complex* \mathbf{X}_v – i.e., the cube complex \mathbf{X} with basepoint v – is equipped with a natural orientation on the 1-skeleton. Indeed, the *initial* 0-cube of the 1-cube c is the 0-cube lying in $v(H)$ and the *terminal* 0-cube of c is the 0-cube lying in $\mathbf{X} - v(H)$. The hyperplanes H, H' of \mathbf{X}_v *contact with respect to* v , denoted $H \perp_v H'$, if H and H' are dual to 1-cubes c, c' , respectively, such that c and c' have the same initial 0-cube. Note that $H \perp_v H'$ only if $H \perp H'$. Also note that if H crosses H' in a 2-cube s , then at least one 0-cube of s is initial in both of its incident 1-cubes in s , and hence, if H and H' cross, then $H \perp_v H'$. On the other hand, if H osculates with H' , then $H \not\perp H'$ if and only if neither of H nor H' separates the other from v .

3.3. Median graphs and parallelism of edges. Let G be a connected graph and let d be the standard path-metric on G (i.e. each edge of G has length 1 and $d(u, v)$ counts the number of edges in a shortest path joining the vertices u and v). The *interval* $I(u, v)$ is the set of all points $x \in G$ such that $d(u, v) = d(x, u) + d(x, v)$. The graph G is a *median graph* if for all triples of vertices $u, v, w \in G$, the set

$$m(u, v, w) = I(u, v) \cap I(u, w) \cap I(v, w)$$

contains exactly one point, also denoted $m(u, v, w)$, called the *median* of u, v, w .

The induced subgraph $G(S)$ of G generated by the set S of vertices is *convex* if for all $u, v \in S$, the interval $I(u, v) \subset G(S)$. The subgraph $G(S)$ is *gated* if for each vertex v of G , there exists a unique vertex $v' = g(v) \in S$, called the *gate of v in S* such that $d(v, y) = d(v, v') + d(v', y)$ for all $y \in S$. Each convex set of vertices of G is gated. A *halfspace* is a convex subset H such that $G - H$ is also convex, and the pair $(H, G - H)$ is a *convex split*.

The relation Θ is defined on the set of edges of G as follows: Θ is the *Djoković-Winkler relation* (“parallelism”) defined as follows. If uv and xy are edges of G , then $(uv, xy) \in \Theta$ if and only if

$$d(u, x) + d(v, y) \neq d(u, y) + d(v, x).$$

Equivalently, Θ is the transitive closure of the “opposite” relation: uv and xy are *opposite edges of a 4-cycle* if $uvxy$ is a 4-cycle in G (see [EFO07, IK00]). The equivalence class $\Theta(uv)$ defines a convex cut-set $\mathbf{A}(uv)$ of G , and complement $\mathbf{B}(uv) = G - \mathbf{A}(uv)$ is also convex. The class $\Theta(uv)$ therefore determines a convex split $(\mathbf{A}(uv), \mathbf{B}(uv))$ of G [Mul80, vdV93]. Conversely, for each convex split (\mathbf{A}, \mathbf{B}) , there exists at least one edge uv such that $\mathbf{A} = \mathbf{A}(uv)$ and $\mathbf{B} = \mathbf{B}(uv)$. The convex splits $(\mathbf{A}_1, \mathbf{B}_1)$ and $(\mathbf{A}_2, \mathbf{B}_2)$ are *incompatible* if and only if each of the sets $\mathbf{A}_1 \cap \mathbf{A}_2$, $\mathbf{A}_1 \cap \mathbf{B}_2$, $\mathbf{B}_1 \cap \mathbf{A}_2$, $\mathbf{B}_1 \cap \mathbf{B}_2$ is nonempty.

The resemblance to the definition of crossing hyperplanes, and the use of the notation Θ for the set of 1-cubes dual to a hyperplane of a CAT(0) cube complex is not accidental. For each median graph G , there exists a CAT(0) cube complex \mathbf{X} whose 1-skeleton is G , and the hyperplanes of \mathbf{X} correspond bijectively to convex splits of G : the equivalence class $\Theta(uv)$ of the edge uv of G is precisely the set of 1-cubes dual to the hyperplane H of \mathbf{X} that crosses the 1-cube uv . Conversely, the 1-skeleton $G(\mathbf{X})$ of the CAT(0) cube complex \mathbf{X} is a median graph, and the hyperplanes of \mathbf{X} correspond in the same way to the convex splits of $G(\mathbf{X})$ (see [Che00]).

There is thus a perfect analogy between the following notions about median graphs and the corresponding notions about CAT(0) cube complexes. If S is a set of vertices, and $H = (\mathbf{A}, \mathbf{B})$ is a convex split of G , then H *crosses S* if there exist $u, v \in S$ with $u \in \mathbf{A}$ and $v \in \mathbf{B}$. In particular, the crossing of two hyperplanes H, H' of \mathbf{X} corresponds to incompatibility of the corresponding convex splits of $G = G(\mathbf{X})$. Likewise, separation of subgraphs of G by a convex split corresponds to separation of those subgraphs by the corresponding hyperplane.

Choosing a base vertex v of G , we define an orientation of all edges. Let xy be an edge such that $d(x, v) \leq d(y, v)$. Then $m = m(v, x, y) = x$ since $d(x, y) = d(m, x) + d(m, y) = 1$ and hence x is strictly closer than y to v . Let x be the initial vertex of xy and y the terminal vertex. Note that if $uv \in \Theta(xy)$, and the terminal vertex x lies in the halfspace $\mathbf{A}(xy)$ of the corresponding convex split, then the terminal vertex u of the edge uv also lies in $\mathbf{A}(xy)$, and hence each convex split of G is oriented. If Θ_1 and Θ_2 are parallelism classes of edges, we write $\Theta_1 \perp_v \Theta_2$ if either the corresponding convex splits are incompatible, or if there exist edges $xy_1 \in \Theta_1$ and $xy_2 \in \Theta_2$ such that x is the initial vertex of xy_1 and xy_2 . Note that the hyperplanes H, H' of \mathbf{X} satisfy $H \perp_v H'$ if and only if $\Theta(H) \perp_v \Theta(H')$.

3.4. Contact and crossing graphs. Let \mathbf{X} be a CAT(0) cube complex and let $G(\mathbf{X}) = \mathbf{X}^{(1)}$ be the corresponding median graph. Let \mathcal{H} be the set of hyperplanes of \mathbf{X} or, equivalently, the set of parallelism classes of edges in $G(\mathbf{X})$. The contact graph of \mathbf{X} was defined in [Hag11] as a modification of the “crossing graph” – the intersection graph of the set \mathcal{H} of hyperplanes in \mathbf{X} – previously studied by Bandelt, Dress, Eppstein, Niblo, Roller, van de Vel and others (see [Rol98] and [Nib04]). While any simplicial graph is the crossing graph of some CAT(0) cube complex, the class of graphs that arise as contact graphs is quite restricted: contact graphs are connected and quasi-isometric to trees [Hag11].

Definition 1 (Contact graph, crossing graph). The *contact graph* $\Gamma(\mathbf{X})$ of \mathbf{X} is the graph whose vertex set is the set \mathcal{H} , with an edge joining the vertices H_1 and H_2 if $H_1 \perp H_2$. We use the same notation for a vertex of $\Gamma(\mathbf{X})$ as for its corresponding hyperplane, and we use the notation $H_1 \perp H_2$ for the (closed) edge of $\Gamma(\mathbf{X})$ joining the contacting hyperplanes H_1 and H_2 .

The *crossing graph* $\Gamma_{\#}(\mathbf{X})$ is the subgraph of $\Gamma(\mathbf{X})$ obtained by deleting each open edge corresponding to an osculating pair of hyperplanes, so that H_1 and H_2 are adjacent in $\Gamma_{\#}(\mathbf{X})$ if and only if they cross.

Given hyperplanes $U, V \in \mathcal{H}$, we denote by $\rho(U, V)$ the distance in $\Gamma(\mathbf{X})$ from U to V .

Likewise, for a pointed CAT(0) cube complex \mathbf{X}_v or pointed median graph $G(\mathbf{X})_v$, we define the pointed contact graph, introduced in [Che11], as follows.

Definition 2 (Pointed contact graph). Then *pointed contact graph* $\Gamma_v(\mathbf{X})$ is the subgraph of $\Gamma(\mathbf{X})$ defined as follows: the vertex set of $\Gamma_v(\mathbf{X})$ is the set \mathcal{H} of hyperplanes of \mathbf{X} , and H and H' are joined by an edge of $\Gamma_v(\mathbf{X})$ if and only if $H \perp_v H'$.

Pointed contact graphs are used in our applications to the nice labeling problem. Observe that if H and H' cross, then the intersection of their carriers contains a 2-cube s , one of whose four 0-cubes must be initial in the incident 1-cubes dual to H and H' . Hence $\Gamma_{\#}(\mathbf{X}) \subseteq \Gamma_v(\mathbf{X}) \subseteq \Gamma(\mathbf{X})$.

3.5. Event structures, nice labeling and the associated cube complexes. The following is an informal summary of the relationship between (pointed) contact graphs of CAT(0) cube complexes and median graphs and nice labelings of event structures, following the treatment given in [Che11].

An *event structure*¹ is a triple $\mathcal{E} = (E, \leq, \smile)$, where E is a set of *events*, \leq is a partial order on E , called *causal dependency*, and \smile is a symmetric, irreflexive binary relation on E called *conflict*. Additionally, \mathcal{E} is *finitary*, which is to say that for all $e, e' \in E$, there exist finitely many $e'' \in E$ such that $e \leq e'' \leq e'$.

The events e and e' are *concurrent*, denoted $e \frown e'$, if they are incomparable in the partial ordering \leq and $e \not\smile e'$. A conflict $e \smile e'$ is *minimal* if there does not exist $e'' \notin \{e, e'\}$ such that e'' precedes one of e and e' in \leq and is in conflict with the other. The events e and

¹Also called a *coherent event structure* or an *event structure with binary conflict*.

e' are *independent* (or *orthogonal*) if they are either concurrent or in minimal conflict. An *independent set* is a set of pairwise independent events in E . The *degree* of E is the maximum cardinality of an independent set in E .

In [RT91], Rozoy and Thiagarajan formulated the *nice labeling problem* for event structures. A *labeling* is a map $\lambda : E \rightarrow \Lambda$, where Λ is some alphabet, and λ is a *nice labeling* if $\lambda(e) \neq \lambda(e')$ whenever e and e' are independent. Solving the nice labeling problem for \mathcal{E} entails constructing a nice labeling λ such that Λ is finite. More quantitatively, one asks, given a class of event structures E , whether there exists a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that, for an event structure \mathcal{E} of degree n in the given class, there exists a nice labeling of \mathcal{E} with $|\Lambda| \leq f(n)$. The first author answered this question in the negative in [Che11] when the class in question is the class of event structures of degree at least five. Theorem 1, however, asserts that the nice labeling conjecture is true for event structures of finite degree that have as their domain a CAT(0) cube complex of dimension at most 2.

The *domain* $\mathcal{D}(\mathcal{E})$ of the event structure \mathcal{E} is defined as follows. A *configuration* C is a subset $C \subseteq E$ of the set of events such that no two elements of C are in conflict, and, if $e \leq e' \in C$ are not in conflict, then $e \in C$. The domain $\mathcal{D}(\mathcal{E})$ is the set of all such configurations C , ordered by inclusion. This construction naturally gives rise to a median graph and an accompanying CAT(0) cube complex associated to \mathcal{E} . Indeed, following [San10], let $G = G(\mathcal{E})$ be the graph whose vertices are the elements of the domain $\mathcal{D}(\mathcal{E})$, with C and C' joined by an edge if and only if $C = C' \cup \{e\}$ for some $e \in E - C$. In this situation, the edge $C'C$ is directed from C' to C . In other words, an event $e \in E$ is viewed as a minimal change from one configuration to another [WN95].

It can be shown [BC93] that G is a median graph, and thus $G = G(\mathbf{X})$, where \mathbf{X} is a CAT(0) cube complex; abusing language slightly, we refer to $G(\mathbf{X})$ or to \mathbf{X} as the *domain* of \mathcal{E} , since these objects are uniquely determined by \mathcal{E} . The hyperplanes of \mathbf{X} correspond bijectively to the events in E . The events e and e' are concurrent if and only if the corresponding hyperplanes cross. In the language of CAT(0) cube complexes, this bijection was recently rediscovered by [AOS12].

Indeed, let C'' be a configuration that does not contain e or e' but such that $C = C'' \cup \{e\}$ and $C' = C'' \cup \{e'\}$ are downward-closed. Then by the definition of concurrency, the configurations C'', C', C , and $C' \cup C$ are the vertices of a 4-cycle in G bounding a 2-cube in \mathbf{X} whose crossing dual hyperplanes correspond to e and e' . On the other hand, if e and e' are in minimal conflict, then C' and C are both adjacent to C'' , and thus C'' has two incident 1-cubes in \mathbf{X} , one dual to each of the hyperplanes e and e' , and hence the corresponding hyperplanes osculate. (The construction of \mathbf{X} from the space of configurations of an event structure is highly reminiscent of the notion of a state complex of a metamorphic robot [AG02, GP07] and of the construction of a cube complex from a wallspace [CN05, Nic04].)

Conversely, each CAT(0) cube complex \mathbf{X} (and thus each median graph $G(\mathbf{X})$ [Che00]), and any fixed base 0-cube $v \in \mathbf{X}$, gives rise to an event structure \mathcal{E} whose events are the hyperplanes of \mathbf{X} [BC93]. Hyperplanes H and H' define concurrent events if and only if they cross, and $H \leq H'$ if and only if $H = H'$ or H separates H' from v . The events defined

by H and H' are in conflict if and only if H and H' do not cross and neither separates the other from v . Thus the events corresponding to H and H' are in minimal conflict if H and H' osculate and neither of H and H' separates the other from v .

Already, from the definition of an event structure, one defines a graph $\mathcal{G}(\mathcal{E})$ whose vertices are the events, with e and e' joined by an edge if and only if e and e' are independent, i.e. if and only if e and e' are concurrent (analogous to crossing hyperplanes) or in minimal conflict (analogous to osculating hyperplanes). Hence $\mathcal{G}(\mathcal{E})$ is a spanning subgraph of the contact graph $\Gamma(\mathbf{X})$ that contains the crossing graph $\Gamma_{\#}(\mathbf{X})$. On the other hand, given a pointed CAT(0) cube complex \mathbf{X}_v , we see that the graph associated to the event structure \mathcal{E}_v is precisely the pointed contact graph $\Gamma_v(\mathbf{X})$. It was noted already in [San10] that a nice labeling of \mathcal{E} corresponds to a colouring of the edges of the corresponding median graph in such a way that edges dual to the same hyperplane receive the same colour, and edges c, c' dual to a pair of hyperplanes H, H' that cross or osculate in a “minimal conflict” way receive different colours. Thus we see that \mathcal{E} admits a nice labeling if the corresponding pointed CAT(0) cube complex has finite chromatic number for its pointed contact graph, and in particular, \mathcal{E} admits a nice labeling if the corresponding contact graph has finite chromatic number. Conversely, if $\Gamma_v(\mathbf{X})$ has infinite chromatic number, then the corresponding event structure \mathcal{E}_v does not admit a nice labeling.

3.6. Disc diagrams. We shall frequently use the technique of minimal-area disc diagrams in the CAT(0) cube complex \mathbf{X} . For a discussion of disc diagrams over cube complexes, using the language and notation closest to that of the present paper, see [Hag11] or [Wis]. The results we shall use are summarized below.

A *disc diagram* $D \rightarrow \mathbf{X}$ in the CAT(0) cube complex \mathbf{X} is a combinatorial map, where D is a contractible 2-dimensional cube complex, not necessarily CAT(0), such that D is equipped with a specific embedding in S^2 , so that $S^2 = D \cup E$, where E is a 2-cell. The *boundary path* of D is the combinatorial path $P \rightarrow \mathbf{X}$ is the restriction of $D \rightarrow \mathbf{X}$ to the attaching map of E .

Proposition 1 (Existence of disc diagrams). *Let $P \rightarrow G(\mathbf{X})$ be a closed path. Then there exists a disc diagram $D \rightarrow \mathbf{X}$ whose boundary path is P .*

Let $D \rightarrow \mathbf{X}$ be a disc diagram. The *area* of D is the number of 2-cubes in D . The disc diagram has *minimal area for its boundary path* P if for all disc diagrams D' with boundary path P , the area of D' is at least the area of D . Note that the equivalence relation Θ on the 1-cubes of \mathbf{X} pulls back to an equivalence relation on the 1-cubes of D . Each equivalence class $\Theta(H)$ of 1-cubes in D determines a *dual curve*, defined as follows. If s is a 2-cube of D , and c, c' are opposite 1-cubes of s with $c, c' \in \Theta(H)$, then the *midcube* of s corresponding to H is the ℓ_1 geodesic segment in s joining the midpoint of c to the midpoint of c' . A *dual curve* K is a maximal concatenation of midcubes of 2-cubes in D . The map $D \rightarrow \mathbf{X}$ restricts to a map $K \rightarrow H$, and moreover, the union $N(K)$ of all 2-cubes in D containing constituent midcubes of K – the *carrier* of K – maps to the carrier $N(H)$. A dual curve K is an immersed interval or circle in D .

If K is a dual curve, then an *end* of K is a midpoint p of a 1-cube c such that p is contained in only one constituent midcube of K . K has at most two ends, and each end of K lies on the boundary path P of D . The following proposition states, in the language of [Wis], that a minimal-area disc diagram does not contain a “nongon” or a “monogon” of dual curves, i.e. dual curves begin and end on the boundary path of D , and no dual curve crosses itself.

Proposition 2 (No nongons or monogons). *Let $D \rightarrow \mathbf{X}$ have minimal area for its boundary path P . Then every dual curve K in D is an embedded interval (possibly of length 0), and in particular each 2-cube s of D contains 1-cubes of exactly two distinct equivalence classes in Θ . If $|K| > 0$, then K has exactly two ends on P . If K is a single point, then $K \in P$.*

If $Q \subseteq P$ is a subpath of the boundary path of D , we say that K *emanates from* or *ends on* Q if Q contains a 1-cube whose midpoint lies in K .

Proposition 3 (No bigons). *If D is a minimal-area disc diagram for its boundary path P , and K_1 and K_2 are distinct dual curves in D , then either $K_1 \cap K_2 = \emptyset$, or $K_1 \cap K_2$ consists of a single point. In the latter case, K_1 and K_2 are said to *cross*. If K_1 and K_2 cross, then their corresponding hyperplanes also cross, and in particular K_1 and K_2 map to distinct hyperplanes.*

Propositions 2 and 3 are used implicitly in all of our disc diagram arguments. The situation concerning trigons of dual curves is somewhat more subtle: a trigon of dual curves along the boundary path of D , for the diagrams we consider, in general contradicts minimality of the area of D (except in certain special cases). However, as in the proof of Theorem 1, if \mathbf{X} is at most 2-dimensional, then all trigons of dual curves mapping to distinct hyperplanes are impossible, regardless of minimality of area, since the existence of pairwise-crossing triples of hyperplanes guarantees the presence of 3-cubes.

Definition 3 (Trigon, trigon along the boundary). Let $D \rightarrow \mathbf{X}$ be a disc diagram with boundary path P . If K_1, K_2, K_3 are distinct pairwise crossing dual curves in D , then they form a *trigon of dual curves*, as at left in Figure 1. Let c_1 and c_2 be distinct 1-cubes of P , and let $c_1 Q c_2$ be one of the paths on P subtended by c_1 and c_2 . For $i \in \{1, 2\}$, let K_i be the dual curve in D emanating from c_i . Suppose that K_1 and K_2 cross, and there exists a hyperplane H such that the image of $c_1 Q c_2$ lies in $N(H)$, with neither c_1 nor c_2 dual to H . Then the pair K_1, K_2 forms a *trigon along the boundary of D* . See the middle of Figure 1.

Using *hexagon moves*, one proves the first assertion in the following proposition. The second follows from the fact that $\dim \mathbf{X}$ is bounded below by the cardinality of any set of pairwise-crossing hyperplanes.

Every disc diagram D in this paper has *fixed carriers* in the sense of [Hag11]. This means that there is a fixed collection $\{H_i\}_{i=1}^k$ such that $\partial_p D = P_1 P_2 \dots P_n$, where P_n is a combinatorial geodesic segment in $N(H_i)$. The diagram D is minimal if it has minimal area among all diagrams with boundary path $\partial_p D$, and, fixing the collection $\{H_i\}$, the paths P_i are chosen so as to minimize the area of D among all diagrams thus constructed. Finally, D is chosen among all such minimal-area diagrams in such a way that $|\partial_p D|$ is minimal.

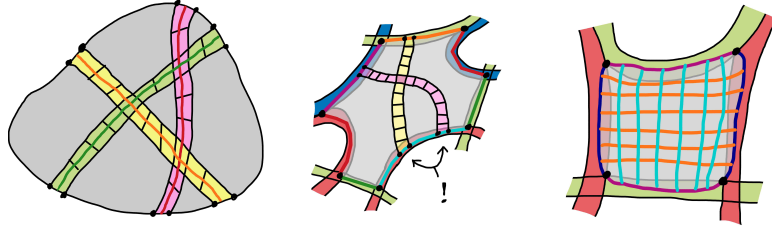


FIGURE 1. At left is a trigon of dual curves, which is in general possible in a minimal-area diagram, but which cannot occur in two dimensions. In the center is a trigon along the boundary, which is always disallowed by minimality of area in our diagrams. At right is a grid.

Proposition 4. *Let D be a diagram with fixed carriers that is minimal in the above sense. Then D contains no trigon K_1, K_2, c_1Qc_2 along the boundary.*

If $\dim \mathbf{X} \leq 2$, then any diagram $D \rightarrow \mathbf{X}$ that has minimal area for its boundary path contains no trigon of dual curves and no trigon along the boundary.

Definition 4 (Grid). Denote by $[0, m]$ the tree with $m + 1$ vertices and m edges, such that each vertex has valence 1 or 2 (i.e. a subdivided line segment). A disc diagram D is a *grid* if $D \cong [0, m] \times [0, n]$ for some m, n .

Remark 1. Note that if \mathbf{X} is 2-dimensional and D is a minimal-area disc diagram in \mathbf{X} , then D is itself a CAT(0) cube complex whose hyperplanes are the dual curves, since any triangle in the link of a 0-cube in D would result in a trigon of dual curves, contradicting Proposition 4.

Moreover, if H_1 and H_2 are hyperplanes represented by dual curves K_1 and K_2 in D , and D is minimal for a set of fixed carriers, and a subtended path $c_1Qc_2 \subset P$ between K_1 and K_2 , with c_i dual to K_i , maps to the carrier of a single hyperplane H whose carrier is one of the fixed carriers of D , then H_1 contacts H_2 if and only if K_1 contacts K_2 . Indeed, by the Helly property described below, $N(H_1) \cap N(H_2) \cap N(H)$ contains a 0-cube p . Consider a path $R_1 \rightarrow N(H_1)$ joining the initial 0-cube of c_1 to p , and a path $R_2 \rightarrow N(H_2)$ joining p to the terminal 1-cube of c_2 . Then $R_1R_2(c_1Qc_2)^{-1}$ bounds a disc diagram E . By choosing E minimal relative to the fixed carriers $N(H_1), N(H_2), N(H)$, we find that we can replace Q by a single 0-cube in $N(H_1) \cap N(H_2) \cap N(H)$, and thus replace D by a lower-area diagram (with a shorter boundary path) and the same set of fixed carriers.

3.7. Isometric embeddings, convexity and the Helly property. We briefly review some notions about isometric embeddings and convex subcomplexes of CAT(0) cube complexes. As usual, the combinatorial map $\mathbf{Y} \rightarrow \mathbf{X}$ is an isometric embedding if the distance between any two points $x, y \in \mathbf{Y}$ (with respect to the wall-metric) is equal to the distance in \mathbf{X} between the images of x and y in \mathbf{X} . We have the following characterization of isometrically embedded subcomplexes of \mathbf{X} .

Proposition 5. *Let $\mathbf{Y} \subseteq \mathbf{X}$ be a subcomplex, and let $\mathcal{H}(\mathbf{Y})$ be the set of hyperplanes crossing \mathbf{Y} . Then $\mathcal{H}(\mathbf{Y})$ is an inseparable set, and for all $H \in \mathcal{H}(\mathbf{Y})$, the intersection $H \cap \mathbf{Y}$ is connected, and $N(H) \cap \mathbf{Y}$ is connected.*

Conversely, let \mathbf{Y} and \mathbf{X} be locally finite $CAT(0)$ cube complexes with hyperplane sets $\mathcal{H}(\mathbf{Y})$ and $\mathcal{H}(\mathbf{X})$ respectively. Suppose there exists an injective graph homomorphism $\phi : \Gamma\mathbf{Y} \rightarrow \Gamma\mathbf{X}$ that is bijective on vertices and sends crossing edges to crossing edges. Suppose moreover that, if U, V, W are hyperplanes of \mathbf{Y} such that V separates U from W , then either $\phi(V)$ separates $\phi(U)$ from $\phi(W)$ or $\phi(V)$ crosses $\phi(U)$ or $\phi(W)$. Then there is an isometric embedding $\mathbf{Y} \rightarrow \mathbf{X}$.

Proof. Suppose $H, H' \in \mathcal{H}(\mathbf{Y})$ and that H'' separates H from H' . Then H'' must separate $H \cap Y$ from $H' \cap Y$, and hence each halfspace of H'' contains a nonempty subspace of Y . Let $y, y' \in H \cap Y, H' \cap Y$ be 0-cubes in distinct halfspaces of H'' . Since Y is isometrically embedded, there exists a geodesic segment $P \rightarrow Y$ joining y and y' , and P must contain a 1-cube dual to H'' . Hence H'' crosses Y and $\mathcal{H}(\mathbf{Y})$ is inseparable.

Suppose now that $H \in \mathcal{H}(\mathbf{Y})$ is a hyperplane such that $N(H) \cap Y$ is disconnected, and let y, y' be 0-cubes in distinct components of $H \cap Y$. Let $P \rightarrow Y$ be a geodesic segment joining y to y' and let $Q \rightarrow N(H)$ be a geodesic segment joining y to y' . Then PQ^{-1} bounds a minimal-area disc diagram $D \rightarrow \mathbf{X}$, and since $|P| = |Q|$, each dual curve in D travels from P to Q . No two dual curves emanating from Q can cross, for otherwise there would be a trigon along the boundary lowering the area of D , and thus $P = Q$. Hence $Q \subset Y \cap N(H)$, and thus y, y' actually belong to the same component of $N(H) \cap Y$. Hence $N(H) \cap Y$ is connected, and thus $H \cap Y$ is also.

To prove the converse, one considers the cube complex \mathbf{Z} dual to the wallspace $(\mathbf{X}^{(0)}, \phi(\mathcal{H}(\mathbf{Y})))$ and verifies that each 0-cube of \mathbf{Z} (consistent choice of halfspace of each hyperplane in $\phi(\mathcal{H}(\mathbf{Y}))$) extends to a 0-cube of \mathbf{X} – one orients hyperplanes of $\mathcal{H}(\mathbf{X}) - \phi(\mathcal{H}(\mathbf{Y}))$ toward some fixed 0-cube in the carrier of $N(H)$, for $H \in \phi(\mathcal{H}(\mathbf{Y}))$, using inseparability. This yields an isometric embedding $\mathbf{Z} \rightarrow \mathbf{X}$. One then verifies, from the fact that ϕ is a graph homomorphism, that \mathbf{Z} and \mathbf{Y} are isomorphic. See Proposition 2.16 of [Hag11]. \square

Note also that, since $G(\mathbf{X})$ is an isometric subspace of \mathbf{X} , an isometric embedding $\mathbf{X} \rightarrow \mathbf{Y}$ restricts to an isometric embedding $G(\mathbf{X}) \rightarrow \mathbf{Y}$. Proposition 5 yields the equality $\tau(\mathbf{X}) = \chi(\Gamma_{\#}(\mathbf{X}))$:

Corollary 1. *The $CAT(0)$ cube complex \mathbf{X} (and hence $G(\mathbf{X})$) isometrically embeds in a Cartesian product \mathbf{Y} of at most k trees if and only if $\chi(\Gamma_{\#}(\mathbf{X})) \leq k$.*

Proof. Let $\mathbf{Y} = \prod_{i=1}^k \mathbf{T}_i$. Then $\Gamma_{\#}(\mathbf{Y})$ is the join of k totally disconnected graphs Γ_i , where Γ_i is the crossing graph of \mathbf{T}_i . Suppose there is an isometric embedding $\psi : \mathbf{X} \rightarrow \mathbf{Y}$. Then there is an induced graph homomorphism $\Gamma_{\#}(\mathbf{X}) \rightarrow \Gamma_{\#}(\mathbf{Y})$. We colour $\Gamma_{\#}(\mathbf{X})$ by assigning to each hyperplane H the colour i corresponding to the unique subgraph Γ_i containing the image of H in $\Gamma_{\#}(\mathbf{Y})$. If H and H' cross, then their images in \mathbf{Y} also cross, and hence belong to distinct factors Γ_i . Hence H and H' receive distinct colours, and this is thus a correct colouring in k colours, i.e. $\chi(\Gamma_{\#}(\mathbf{X})) \leq k$.

Conversely, let $c : \mathcal{H} \rightarrow \mathcal{K}$ be a correct colouring of $\Gamma_{\#}(\mathbf{X})$ in the set \mathcal{K} of k colours. For each $i \in \mathcal{K}$, let $\mathcal{H}_i = c^{-1}(i)$ be the set of hyperplanes with the colour i . For each $i \in \mathcal{K}$, let \mathbf{T}_i be the CAT(0) cube complex dual to the wallspace $(\mathbf{X}^{(0)}, \mathcal{H}_i)$. Since c is a correct colouring of the crossing graph, no two elements of \mathcal{H}_i cross, and thus \mathbf{T}_i is a tree. Let $\mathbf{Y} = \prod_{i=1}^k \mathbf{T}_i$. Each hyperplane of \mathbf{Y} is of the form

$$\mathbf{H}(H, i) = \mathbf{T}_1 \times \dots \times \mathbf{T}_{i-1} \times H \times \mathbf{T}_{i+1} \times \dots \times \mathbf{T}_k$$

for some $H \in \mathcal{H}_i$ with $1 \leq i \leq k$. Moreover, $\mathbf{H}(H, i)$ and $\mathbf{H}(H', j)$ are distinct if $H \neq H'$ and cross if and only if $i \neq j$. Furthermore, each $H \in \mathcal{H}$ gives rise to a hyperplane of this form, by construction. Hence the identification $\mathcal{H}_i \ni H \mapsto \mathbf{H}(H, i)$ is a bijection yielding a graph homomorphism $\Gamma_{\#}(\mathbf{X}) \rightarrow \Gamma_{\#}(\mathbf{Y})$ satisfying the separation hypothesis of Proposition 5. Thus, by Proposition 5, there is an isometric embedding $\mathbf{X} \rightarrow \mathbf{Y}$. \square

Convexity of a subcomplex $\mathbf{Y} \subseteq \mathbf{X}$ is characterized as follows: the subcomplex \mathbf{Y} is convex if and only if, whenever H and H' are hyperplanes that cross \mathbf{Y} , either H and H' do not contact or $N(H) \cap N(H') \cap \mathbf{Y} \neq \emptyset$ and, if H and H' cross, then \mathbf{Y} contains a 2-cube representing this crossing. In particular, the contact graph of \mathbf{Y} is an induced subgraph of $\Gamma(\mathbf{X})$ whose vertex set is inseparable. From the point of view of median graphs, one verifies that, since \mathbf{Y} is gated if it is convex, if $\Theta(H)$ and $\Theta(H')$ contain 1-cubes c and c' with a common 0-cube v , then the gate of v in \mathbf{Y} must lie in $N(H) \cap N(H')$.

Note also that \mathbf{X} enjoys the *Helly property*: if Y_1, Y_2, \dots, Y_n are convex subcomplexes of \mathbf{X} such that $Y_i \cap Y_j \neq \emptyset$ for $i \neq j$, then $\bigcap_{i=1}^n Y_i \neq \emptyset$. This follows from the fact that convex subgraphs of $G(\mathbf{X})$ are gated, or from a simple disc diagram argument ([Hag11]).

3.8. Hyperplane-distance. Let U be a fixed hyperplane of \tilde{X} . In our applications, U will be the central hyperplane of a specified sphere in $\Gamma(\mathbf{X})$ of radius 2.

Definition 5 (Hyperplane-distance). For each hyperplane H , the set $\mathfrak{J} = \{V_1, \dots, V_n\}$ is a *separating chain* for H if

- (1) Each $V_i \in \mathfrak{J}$ separates H from U .
- (2) If $V_i, V_j \in \mathfrak{J}$ are separated by a hyperplane V , then $V \in \mathfrak{J}$.
- (3) The hyperplanes in \mathfrak{J} are pairwise non-crossing.

These properties ensure that the halfspaces of the $V_i \in \mathfrak{J}$ can be labeled **A** or **B** so that $\{\mathbf{B}(V_i)\}_{V_i \in \mathfrak{J}}$ is totally ordered by inclusion. Let \mathbb{I} be the set of all separating chains for H . The *hyperplane-distance* of H (with respect to U) is

$$d(H) = \min\{|\mathfrak{J}| : \mathfrak{J} \in \mathbb{I}\}.$$

If C is a collection of hyperplanes (or a subgraph of $\Gamma(\mathbf{X})$), we let $D(C) = \sum_{H \in C} d(H)$.

Note that $d_{G(\mathbf{X})}(N(H), N(U))$ counts the hyperplanes that separate U from H , since carriers are convex, and that $d(H)$ is bounded above by this quantity. Note also that $d(H) = 0$ if and only if $H \perp U$.

Lemma 1. *Let H be a hyperplane with $\rho(U, H) = 2$ and let F be a hyperplane such that $U \perp F \perp H$. Then there exists a path $P \rightarrow N(F)$ such that $d_{G(\mathbf{X})}(N(H), N(U)) = |P|$.*

Proof. Let $P \rightarrow N(F)$ be a shortest combinatorial path joining a 0-cube $a \in N(F) \cap N(H)$ to a 0-cube $b \in N(F) \cap N(U)$. Let $Q \rightarrow G(\mathbf{X})$ be a path realizing the distance from $N(H)$ to $N(U)$, with endpoints $c \in N(H)$ and $d \in N(U)$. Let $A \rightarrow N(H)$ be a geodesic segment joining a to c and let $B \rightarrow N(U)$ be a geodesic segment joining b to d . Then there exists a minimal-area disc diagram $D \rightarrow \mathbf{X}$ with boundary path $PBQ^{-1}A^{-1}$. Every dual curve in D emanating from P ends on Q , since otherwise there is a trigon of dual curves along the boundary path of D , since each of the paths P, A, B maps to the carrier of a hyperplane. Thus $|P| \leq |Q|$ and hence $|P| = |Q| = d_{G(\mathbf{X})}(N(H), N(U))$ by minimality of $|Q|$. \square

Remark 2. Note $d_{G(\mathbf{X})}(N(H), N(U)) > 0$ if and only if $\rho(U, H) \geq 2$. In our applications, Lemma 1 is applied in such a way that P lies in the *father* of H , defined below.

3.9. Footprints and imprints. In this section, we suppose that the CAT(0) cube complex \mathbf{X} is 2-dimensional and in particular that each hyperplane of \mathbf{X} is a tree. The carrier $N(H)$ of any hyperplane H is bounded by two disjoint subcomplexes H^+ and H^- which are both isomorphic to H and constitute convex and therefore gated subcomplexes of \mathbf{X} . If V is a hyperplane contacting H , then the *footprint* of H in V is $F(H, V) = N(V) \cap N(H)$. Any footprint is gated as the intersection of two gated subcomplexes. If V and H are osculating, then $F(H, V)$ is completely contained in H^+ or in H^- . On the other hand, if V and H are crossing, then $F(H, V)$ contains the union of two isomorphic subcomplexes $F^+(H, V) = N(V) \cap H^+$ and $F^-(H, V) = N(V) \cap H^-$, each of which is a 1-cube, and $F(H, V)$ is a single 2-cube, since \mathbf{X} is 2-dimensional. We call the projection of the footprint $F(H, V)$ on V the *imprint* of H on V and denote it by $J(H, V)$. Denote by $\mathcal{F}(V)$ and $\mathcal{J}(V)$ the set systems consisting of all footprints $F(H, V)$ and of all imprints of all hyperplanes H such that $H \perp V$. We emphasize that, if H and H' are distinct hyperplanes that contact V , it may happen that $F(H, V)$ and $F(H', V)$ denote the same subcomplex of $N(V)$, but we regard them as distinct elements of $\mathcal{F}(V)$.

We begin our discussion of footprints and imprints with a consequence of the Helly property for hyperplanes.

Lemma 2. *Let H', H'', V be hyperplanes such that $H' \perp V$ and $H'' \perp V$. Then $H' \perp H''$ if and only if $F(H', V) \cap F(H'', V) \neq \emptyset$. If $H' \perp H''$, then $J(H', V) \cap J(H'', V) \neq \emptyset$.*

Proof. If $v \in F(H', V) \cap F(H'', V)$, then by definition of footprints we conclude that $v \in N(H') \cap N(H'')$, yielding $H' \perp H''$. Conversely, if $H' \perp H''$, then there exists $v \in N(H') \cap N(H'')$. Let v_0 be the gate of v in $N(V)$. Pick $x' \in N(V) \cap N(H')$ and $x'' \in N(V) \cap N(H'')$. Since the carriers $N(H')$ and $N(H'')$ are convex and $x', v \in N(H')$, $x'', v \in N(H'')$, and $v_0 \in I(v, x') \cap I(v, x'')$, we conclude that $v_0 \in N(H') \cap N(H'')$. Since v_0 also belongs to $N(V)$, this implies that $v_0 \in F(H', V) \cap F(H'', V)$. Finally, the projection v'_0 of v_0 in V belongs to the imprints $J(H', V)$ and $J(H'', V)$. \square

Lemma 3. *For a hyperplane V , any vertex v of the 1-skeleton of $N(V)$ belongs to at most Δ footprints from the family $\mathcal{F}(V)$. In particular, $\delta(\mathcal{F}(V)) \leq \Delta$ and $\delta(\mathcal{J}(V)) \leq 2\Delta$.*

Proof. The degree of v in $G(\mathbf{X})$ and therefore in the 1-skeleton of $N(V)$ is at most Δ . Consider the set \mathcal{H}_v of all hyperplanes H such that $v \in F(H, V)$. If $H \in \mathcal{H}_v$ crosses the hyperplane V , then the equivalence class $\Theta(H)$ of H contains an edge e_H incident to the vertex v and belonging to $N(V)$. Analogously, if the hyperplanes $H \in \mathcal{H}_v$ and V osculate, then any vertex of $F(H, V)$, in particular v , is incident to an edge e_H of $\Theta(H)$ (in this case e_H does not belong to $N(V)$). Two edges $e_H, e_{H'}$ defined by two different hyperplanes $H, H' \in \mathcal{H}_v$ are different because they belong to two different equivalence classes of the relation Θ . Thus $|\mathcal{H}_v| \leq \Delta$, establishing that $\delta(\mathcal{F}(V)) \leq \Delta$. Since each vertex v_0 of the tree V is the image of two vertices of the 1-skeleton of $N(V)$, which belong to at most Δ footprints each, v_0 belongs to at most 2Δ imprints. \square

Proposition 6. $\chi(\mathcal{F}(V)) \leq \chi(\mathcal{J}(V)) \leq 2\Delta$.

Proof. The inequality $\chi(\mathcal{F}(V)) \leq \chi(\mathcal{J}(V))$ is obvious because two intersecting footprints give rise to two intersecting imprints. It is well-known (see, for example, [Gol80], [GLB03]) that if \mathcal{F} is a family of subtrees of a tree T , then $\chi(\mathcal{F}) = \omega(\mathcal{F})$ and, since \mathcal{F} satisfies the Helly property, $\omega(\mathcal{F}) = \delta(\mathcal{F})$. Since $\mathcal{J}(V)$ is a family of subtrees of the tree V and $\delta(\mathcal{J}(V)) \leq 2\Delta$ by Lemma 3, we conclude that $\chi(\mathcal{J}(V)) \leq 2\Delta$. \square

4. CANONICAL PATHS, GRANDFATHERS, AND THE WEAK COMBING PROPERTY

Choose, once and for all, an arbitrary but fixed *base hyperplane* $H_0 \in \mathcal{H}$. For any $H \in \mathcal{H}$, the *grade* of H is $g(H) = \rho(H, H_0)$.

Definition 6 (Ball, sphere, cluster). For each $r \geq 0$, the (*full*) *ball* $B_r := B_r(H_0)$ is the full (i.e., induced) subgraph of $\Gamma(\mathbf{X})$ generated by the set of hyperplanes H with $g(H) \leq r$. The (*full*) *sphere* $S_r := S_r(H_0)$ is the full subgraph of $\Gamma(\mathbf{X})$ generated by the set of hyperplanes H with $g(H) = r$.

Let $H, H' \in S_r$ be hyperplanes. Then $H \sim H'$ if and only if there exists a path P in $\Gamma(\mathbf{X})$ joining H to H' such that every vertex of P corresponds to a hyperplane of grade at least r . This defines an equivalence relation on the grade- r hyperplanes. An equivalence class of hyperplanes of grade r is called a *grade- r cluster*.

The notion of a *realization* allows us to translate statements about paths in $\Gamma(\mathbf{X})$ into statements about paths in \mathbf{X} . More specifically, note that if $H_0 \perp H_1 \perp H_2$ is a path in $\Gamma(\mathbf{X})$, then we have a path $c_0 P c_2$ in \mathbf{X} , where c_0, c_2 are 1-cubes dual to H_0 and H_2 , respectively, and $P \rightarrow N(H_1)$ is a combinatorial path joining a 0-cube of $N(H_0) \cap N(H_1)$ to a 0-cube of $N(H_1) \cap N(H_2)$.

Definition 7 (Realization, canonical path). Let $H_0 \perp H_1 \perp H_2$ be a path in Γ . An edge-*realization* of $H_0 \perp H_1 \perp H_2$ is a combinatorial geodesic $P \rightarrow N(H_1)$ that joins $N(H_0)$ to $N(H_2)$. If $\gamma = H_0 \perp H_1 \perp H_2 \perp \dots \perp H_r$ is an embedded path in Γ , then a *realization* of γ is a path $\mathbf{R}(\gamma) = R_1 R_2 \dots R_{r-1}$, where each R_i is a realization of the path $H_{i-1} H_i H_{i+1}$.

Let $\gamma = H_0 \perp H_1 \perp \dots \perp H_r = H$ be a geodesic path in $\Gamma(\mathbf{X})$. The *weight* $\|\gamma\|$ of γ is the ordered r -tuple $(|R_{r-1}|, |R_{r-2}|, \dots, |R_1|)$, where $\mathbf{R}(\gamma) = R_1 R_2 \dots R_{r-1}$ is a realization

such that the previous r -tuple is minimal in the lexicographic order as $\mathbf{R}(\gamma)$ varies among realizations of γ .

A path $\gamma(H) = H_0 \triangleleft H_1 \triangleleft \dots \triangleleft H_r = H$ is a *canonical path* for H if $\|\gamma(H)\|$ is minimal (in the lexicographic order) among all paths γ of $\Gamma(\mathbf{X})$ joining H_0 to H . The hyperplane $f^2(H) = H_{r-2}$ is called the *grandfather* of H (with respect to $\gamma(H)$).

Figure 3 contains heuristic pictures of realizations. Given a grade- r hyperplane $H = H_r$, there are in general many canonical paths joining H_0 to H_r . Figure 2 shows a grade-3 hyperplane in a CAT(0) cube complex, and two distinct canonical paths, along with their realizations. Figure 2 also shows that, in general, a given path in \mathbf{X} may realize many paths in $\Gamma(\mathbf{X})$.

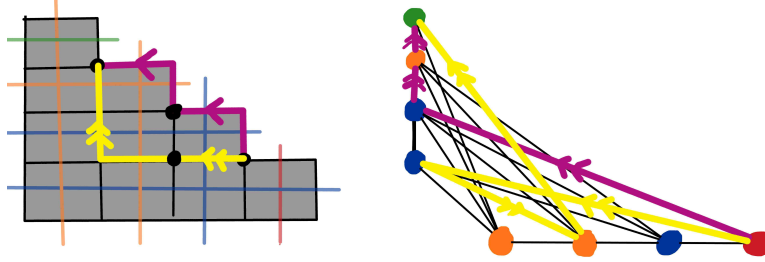


FIGURE 2. \mathbf{X} is shown at left and the contact graph $\Gamma(\mathbf{X})$ at right. Arrowed paths in \mathbf{X} are least-weight realizations of the correspondingly-arrowed paths in $\Gamma(\mathbf{X})$.

Proposition 7. *Let H, H' be two hyperplanes belonging to a common grade- r cluster of $\Gamma(\mathbf{X})$, with $r \geq 2$, and let $\gamma(H), \gamma(H')$ be two canonical paths, respectively joining H_0 to $H = H_r$ and to $H' = H'_r$. Then the grandfathers $f^2(H), f^2(H')$ of H and H' in $\gamma(H)$ and $\gamma(H')$ either coincide or contact, i.e., either $H_{r-2} = H'_{r-2}$ or $H_{r-2} \triangleleft H'_{r-2}$.*

Proof. First, note that the claim is obviously true for $r = 2$, since in that case $H_{r-2} = H'_{r-2} = H_0$, so assume $r \geq 3$ and assume that $H_{r-2} \neq H'_{r-2}$.

The disc diagram D : Let $\mathbf{R}(\gamma(H)) = R_1 R_2 \dots R_{r-1}$ and $\mathbf{R}(\gamma(H')) = R'_1 R'_2 \dots R'_{r-1}$ be least-weight realizations of $\gamma(H)$ and $\gamma(H')$ respectively, so that $R_i \rightarrow N(H_i)$ and $R'_i \rightarrow N(H'_i)$ are combinatorial geodesics for each i . Let $P_0 \rightarrow N(H_0)$ be a combinatorial geodesic joining the initial 0-cubes of R_1 and R'_1 .

Since H and H' belong to the same grade- r cluster, then by definition there exists a shortest path $H = V^0 \triangleleft V^1 \triangleleft V^2 \dots \triangleleft V^k = H'$ joining H to H' , and $g(V_i) \geq r$ for $0 \leq i \leq k$. Hence there is a concatenation $Q = Q_0 Q_1 \dots Q_k$, where $Q_i \mapsto N(V_i)$ is a combinatorial geodesic, joining the terminal 0-cube of R_{r-1} to that of R'_{r-1} . Hence we have a closed, piecewise-geodesic path

$$A = \left(\prod_{i=1}^{r-1} R_i \right) Q \left(\prod_{i=1}^{r-1} R'_i \right)^{-1} P_0^{-1} \rightarrow G(\mathbf{X}).$$

Let $D \rightarrow \mathbf{X}$ be a minimal-area disc diagram with boundary path A . This notation is illustrated in Figure 3.

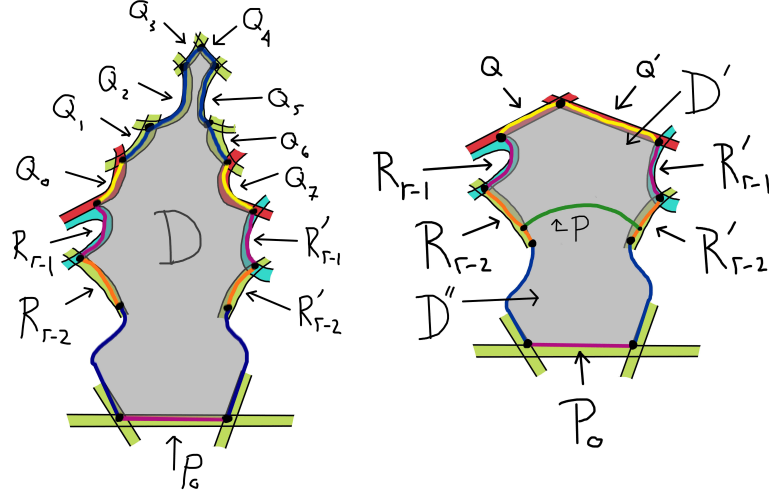


FIGURE 3. The disc diagram D in the proof of Proposition 7 is shown at left. The hyperplane carriers containing the various named subpaths of the boundary path of D are shown. At right is the same diagram, drawn for simplicity in the case where H and H' contact, showing the path P and the resulting subdiagrams D' and D'' .

The path P of $G(\mathbf{X})$ and the subdiagrams D' and D'' : By Lemma 4, there exists a combinatorial path $P \hookrightarrow D$ whose endpoints lie on R_{r-2} and R'_{r-2} , with the property that every dual curve in D crosses P at most once, and no dual curve that crosses P emanates from R_{r-2} or R'_{r-2} . Note that P separates D into two disc diagrams, i.e. $D = D' \cup_P D''$, where D' is the subdiagram containing Q and D'' is the subdiagram containing P_0 , as shown at right in Figure 3.

Analysis of D' : Let K be a dual curve in D' emanating from P and mapping to a hyperplane W . Then there is a dual curve L in D such that $L \cap D' = K$. Since no dual curve crosses more than one 1-cube of P , and no dual curve crossing P ends on R_{r-2} or R'_{r-2} , the dual curve L has exactly one end on the boundary path of D' , i.e. on R_{r-1} , R'_{r-1} or Q , and one end on the boundary path of D'' , on P_0 or R_i or R'_i , with $i \leq r-3$.

Since $r \geq 3$, the end of L on the boundary path of D'' cannot be on P_0 or on R_i or R'_i with $i < r-3$, for otherwise W would contact H_i or H'_i , with $i < r-3$, and also contact H_{r-1} or H'_{r-1} or V^i , contradicting the fact that canonical paths are geodesics in $\Gamma(\mathbf{X})$. Similarly, L cannot end on Q , and hence L travels from R_{r-3} to R'_{r-1} or to R_{r-1} , or, when $r = 3$, from P_0 to R_{r-1} or R'_{r-1} , as shown in Figure 4.

Let S be the path on the carrier $N(L)$ of L that is isomorphic to L and is separated from R'_{r-3} by L . Note that the 1-cube of R'_{r-1} dual to L cannot be the terminal 1-cube of R_{r-1} . Indeed, the hyperplane W has grade at most $r-2$ since L emanates from R_{r-3} ,

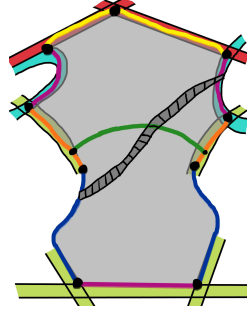


FIGURE 4. The path P and the carrier of a dual curve L that crosses P .

and hence W cannot contact the grade- r hyperplane V' . On the other hand, the 0-cube of S on R'_{r-1} is the terminal 0-cube of a 1-cube contained in R'_{r-1} , and hence the subpath $S'_{r-1} \subset R'_{r-1}$ subtended by S and Q' satisfies $|S'_{r-1}| < |R'_{r-1}|$. We thus have a path $H_0 \uplus H_1 \uplus \dots \uplus H_{r-3} \uplus W \uplus H'_{r-1} \uplus H'$. This path has weight at most

$$\begin{aligned} (|S'_{r-1}|, |S|, \dots) &< (|R'_{r-1}|, |R'_{r-2}|, |R'_{r-3}|, \dots) \\ &= ||\mathbf{R}(\gamma(H'))|| \end{aligned}$$

since $|S'_{r-1}| < |R'_{r-1}|$. This contradicts that $\phi(H')$ is a canonical path.

Note that, were L to travel from R_{r-3} to R_{r-1} , then, as in Figure 5, we would have $|S| = |R_{r-2}|$ and thus W could replace H_{r-2} in $\phi(H)$, leading to a lower-weight path, contradicting the fact that $\gamma(H)$ is canonical. Indeed, the subdiagram between S , R_{r-2} and the subtended

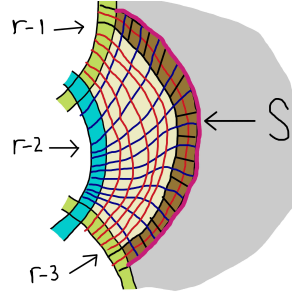


FIGURE 5. When L travels from R_{r-3} to R_{r-1} , we obtain a lower-weight path.

parts of R_{r-3} and R_{r-1} is a grid, since D is of minimal area, and thus $|S| = |R_{r-2}|$. Hence we may assume that L travels from R_{r-3} to R'_{r-1} .

Conclusion: Since any dual curve in D' emanating from P leads to a contradiction either of minimality of the area of D or of the fact that $\gamma(H)$ or $\gamma(H')$ is canonical, we conclude that $|P| = 0$, and hence that $H_{r-2} \uplus H'_{r-2}$. This contact is in fact visible in the diagram D – see Figure 4. \square

Lemma 4. *Using the notation of Proposition 7, there exists a path $P \rightarrow D \rightarrow \mathbf{X}$ such that*

- (1) P joins a 0-cube of R_{r-2} to a 0-cube of R'_{r-2} .
- (2) Each dual curve in D is dual to at most a single 1-cube of P .
- (3) No dual curve in D that crosses P has an end on R_{r-2} or R'_{r-2} .

Proof. Choose a shortest path $P \rightarrow D^{(1)}$ that joins a 0-cube of R_{r-2} to a 0-cube of R'_{r-2} . We first modify P , without affecting its endpoints, so that (2) is satisfied. We then modify P , without increasing its length, until (3) is satisfied.

Modifying P to satisfy (2): Let K be a dual curve in D that is dual to two distinct 1-cubes c, c' of P . Moreover, suppose that K is an innermost such dual curve, in the sense that no two 1-cubes between c and c' on P are dual to the same dual curve. Consider the path T on $N(K)$ traveling from the initial 0-cube of c to the terminal 0-cube of c' . Then T and $cP'c'$ bound a subdiagram E , where P' is the subtended part of P ; see the left picture in Figure 6. Since K is innermost, every dual curve in E travels from P' to T . Indeed, the only other possibility is a dual curve L dual to at least two distinct 1-cubes of T , but that would lead to a bigon between K and L , contradicting minimality of the area of D . Hence $|T| = |cP'c'|$, and we replace P by a new path, with the same endpoints, in which $cP'c'$ is replaced by T . This lowers the number of dual curves that cross P in more than one way, and thus in finitely many such steps we arrive at a choice of P satisfying (2).

Modifying P to satisfy (3): Let C be a dual curve in D that emanates from R_{r-2} and crosses P , as at right in Figure 6. Let $P = P'cP''$, where c is the 1-cube of P dual to C and P' is the subpath of P joining the initial 0-cube of P to the initial 0-cube of c . Let Tc' be the subpath of R_{r-2} between the initial 0-cube of P and the 1-cube c' of R_{r-2} dual to C . Let F be the subdiagram of D bounded by $Tc', P'c$, and S , where S is the shortest path on the carrier of C that joins the endpoints of c and c' .

No dual curve in F emanating from T can end on S , since that would lead to a trigon of dual curves along the boundary path of D and a consequent reduction in area. Hence, as shown in Figure 6, dual curves in F travel from S to P' or from T to P' , or from c to c' . The former type shows that $|S| \leq |P'|$, with equality if and only if $|T| = 0$. We thus have that $|SP''| \leq |P'| + |P''| < |P'cP''| = |P|$, contradicting the assumption that P was a shortest path joining R_{r-2} to R'_{r-2} . Indeed, SP'' has its endpoints on R_{r-2} and R'_{r-2} since c' is a 1-cube of R_{r-2} and P'' is the terminal subpath of P . \square

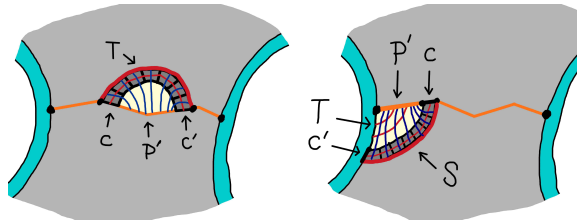


FIGURE 6. Left to right: the subdiagrams E and F of D .

Remark 3. In the case where \mathbf{X} is 2-dimensional, the second part of the proof of Lemma 4 can be simplified slightly, using the fact that a minimal-area disc diagram in a 2-dimensional CAT(0) cube complex is itself a CAT(0) cube complex. Although this ceases to be true in higher dimensions, the proof given above works for arbitrary CAT(0) cube complexes, and indeed the weak combing property established by Proposition 7 and Corollary 2, as well as the bound on the diameters of clusters in $\Gamma(\mathbf{X})$ established in Corollary 3, holds any CAT(0) cube complex.

Applying Proposition 7 to a pair H, H' of contacting hyperplanes of the same grade, we obtain the following property of grandfathers, which is used in colouring Γ :

Corollary 2 (Weak combing). *The grandfathers in canonical paths of two contacting hyperplanes of the same grade either coincide or contact.*

Since the distance in $\Gamma(\mathbf{X})$ from a hyperplane to its grandfather is 2, from Proposition 7 we also immediately obtain:

Corollary 3 (Diameter of clusters). *The diameter of each cluster in the contact graph $\Gamma(\mathbf{X})$ is at most 5.*

Remark 4. Using a more careful (and different) analysis of the disc diagram D , [Hag11] shows that in fact the diameter of each cluster can be bounded by 4. For hyperplane-colouring, we only require that there is a uniform bound on the diameter of clusters.

5. POTENTIAL FATHERS, ITERATED FOOTPRINTS AND IMPRINTS

For a hyperplane U , of grade $r - 2$, and a fixed cluster \mathcal{C} of grade r , denote by $\mathcal{R}(U) = \mathcal{R}(U, \mathcal{C})$ the set of all hyperplanes H in \mathcal{C} such that U is the grandfather of H in a fixed canonical path $\gamma(H)$, i.e., $f^2(H) = U$. As before, U^+ and U^- denote the two copies of U bounding the carrier $N(U)$. For a hyperplane $H \in \mathcal{R}(U)$, denote by $PF(H)$ the set of all hyperplanes V which contact at the same time H and its grandfather U and call any such hyperplane V a *potential father* of H . Let $\mathcal{PF}(H)$ denote the union of carriers $N(V)$, where V varies over the set of potential fathers of H . The *iterated footprint* of H on its grandfather $U = f^2(H)$ is the subcomplex $IF(H, U) = \mathcal{PF}(H) \cap N(U)$. Analogously, the *iterated imprint* $IJ(H, U)$ of H on U is the projection of $IF(H, U)$. Denote by $\mathcal{F}(U)$ and $\mathcal{J}(U)$ the set families consisting of all iterated footprints $IF(H, U)$ and imprints $IJ(H, U)$ taken over all hyperplanes H having U as the grandfather. When the grandfather $U = f^2(H)$ is a fixed hyperplane, we use the notation $IF(H, U) = IF(H)$ for the iterated footprint of H on U .

Lemma 5. *Let $H \in \mathcal{R}(U)$. Then the subcomplex $\mathcal{PF}(H)$ is connected and hence the iterated footprint $IF(H)$ is a connected subcomplex of $N(f^2(H))$. In particular, the iterated imprint $IJ(H)$ is a subtree of U .*

Finally, if $H, H' \in \mathcal{R}(U)$ contact, then $IF(H) \cap IF(H') \neq \emptyset$, and hence the iterated imprints of H and H' in U intersect in a subtree.

Proof. We first show that the set $PF(H)$ is inseparable, i.e. that if $V, V' \in PF(H)$ and V'' separates V' from V , then $V'' \in PF(H)$. Indeed, let $P \rightarrow N(U)$ be a geodesic joining a closest pair of 0-cubes of $F(V, U)$ and $F(V', U)$, let $Q \rightarrow N(H)$ be a shortest geodesic of $N(H)$ joining $F(V, H)$ to $F(V', H)$. Note that P and Q are necessarily disjoint since U and H do not contact. Hence the shortest paths $R, R' \rightarrow N(V), N(V')$ joining the initial and terminal 0-cubes of P, Q , respectively, have length at least 1. Likewise, since V'' separates V and V' , it must separate $F(V, U)$ and $F(V', U)$ and also $F(V, H)$ and $F(V', H)$, and hence V'' crosses P and Q and hence crosses H and U , and thus $V'' \in PF(H)$.

Let D be a minimal-area disc diagram with boundary path $RQ(R')^{-1}P^{-1}$. By minimality of area, dual curves in D travel from R to R' or from P to Q . If C is a dual curve traveling from P to Q , then C maps to a hyperplane V'' that crosses U and H , and hence $V'' \in PF(H)$. Thus the 1-cube $c \subset P$ dual to C lies in $N(V'') \subset IF(H)$, and hence $P \subset IF(H)$. If there is no such dual curve C , then $|P| = 0$ and $N(V) \cap N(V') \neq \emptyset$. Thus $IF(H)$ is connected. The projection $N(U) \rightarrow U$ preserves connectedness, and hence $IJ(H)$ is a connected subtree of U .

Finally, if H, H' contact, then by Lemma 3.8 of [Hag11], either H and H' have a common potential father V , or there exist potential fathers V, V' of H and H' respectively such that $V \perp V'$. In the first case, $F(V, U)$ belongs to the iterated footprint of both H and H' , and in the second case, $F(V, U) \cap F(V', U) \neq \emptyset$ since U is convex. \square

For a hyperplane U , fix once and for all a vertex b^* of U as a root of the tree U . Among the potential fathers of a hyperplane $H \in \mathcal{R}(U)$, pick a hyperplane V whose imprint $J(V, U)$ is closest to b^* , i.e., $d(b^*, IJ(H, U)) = d(b^*, J(V, U)) = \min\{d(b^*, J(V', U)) : V' \in PF(H)\}$ (the distance $d(b^*, J(V, U))$ is measured according to the standard distance between a vertex and a subtree of a tree U). Additionally, if there exist several potential fathers of H whose imprints have the same minimal distance to U , then let V be that potential father for which the imprint $J(H, V)$ is closest to $J(V, U)$. If there are several such hyperplanes V , choose one arbitrarily. Set $f(H) = V$ and call it the *father* of H . The vertex b_H of $IJ(H, U)$ realizing the distance $d(b^*, IJ(H, U))$ is called the *root* of the iterated imprint $IJ(H, U)$. (Note that the path $\gamma^*(H) = H_0 \perp H_1 \perp \dots \perp U = H_{r-2} \perp f(H) \perp H$ obtained from $\gamma(H)$ by replacing the hyperplane H_{r-1} by the father $f(H)$ is a geodesic between H_0 and H in $\Gamma(\mathbf{X})$ but is not necessarily a canonical path.) On $\mathcal{R}(U)$ we define a partial order \prec by setting $H \prec H'$ if and only if $b_H \neq b_{H'}$ and b_H belongs to the unique path of U between b^* and $b_{H'}$ (in this case we will also write $b_{H'} \prec b_H$) and breaking ties arbitrarily when $b_H = b_{H'}$.

Remark 5. We briefly review the logic of the choice of fathers. Recall that we have fixed a base hyperplane H_0 and graded $\Gamma(\mathbf{X})$ with respect to H_0 . We then chose, for each hyperplane H , a canonical path $\gamma(H)$ joining H_0 to H . This choice uniquely determines a grandfather $f^2(H)$ for each hyperplane H of grade at least 2. For any hyperplane U , there is therefore a well-defined set $\mathcal{R}(U)$ containing those hyperplanes H for which, with respect to our fixed choice of canonical paths, $U = f^2(H)$.

We then focus on a single hyperplane U , and fix a base vertex b^* in the tree U . The *father* $f(H)$ of $H \in \mathcal{R}(U)$ is a hyperplane $V = f(H)$ such that $V \perp U, V \perp H$, and no hyperplane V' satisfying these criteria has imprint on U closer to b^* than does V . Note that there could be more than one hyperplane V satisfying these criteria. In this case, we choose *the* father of H arbitrarily among all hyperplanes V with the desired properties. In practice, this arbitrary choice is justified since we shall only use the three given properties of $f(H)$. Having chosen the father of each $H \in \mathcal{R}(U)$, we see that \prec partially orders $\mathcal{R}(U)$.

Lemma 6. *If $H, H' \in \mathcal{R}(U)$ and $H \perp H'$, then one of the following holds:*

- (1) $f(H) = f(H')$.
- (2) $f(H) \perp f(H')$.
- (3) $H \prec H'$ and $f(H')$ contacts a potential father W of H such that W crosses U .
- (4) $H' \prec H$ and $f(H)$ contacts a potential father W' of H' such that W' crosses U .

Proof. The disc diagram D : Let $V = f(H)$ and $V' = f(H')$ and suppose that $V \neq V'$ and that V and V' do not contact. Following the proof of Proposition 7, construct a disc diagram $D \rightarrow \mathbf{X}$ as follows. Let $P \rightarrow N(U)$ be a combinatorial path joining $a \in F(V, U)$ to $c \in F(V', U)$, where a and c are chosen to be the preimages in $N(U)$ of the roots b_H and $b_{H'}$, respectively, of the iterated imprints $IJ(H, U)$ and $IJ(H', U)$. Since V does not contact V' , we have $a \neq c$ and hence $|P| \geq 1$.

Let $R, R' \rightarrow N(V), N(V')$ join a (respectively, c) to a closest 0-cube of $N(H)$ (respectively, $N(H')$). Let $Q, Q' \rightarrow N(H), N(H')$ be shortest geodesics such that $PR'Q'QR^{-1}$ is a closed path, and let D be a minimal-area disc diagram for that path; see the left side of Figure 7. There is at least one dual curve C emanating from P , and C cannot end on R or on R' since that would lead to a trigon of dual curves along the boundary path of D , contradicting minimality of area. Thus C ends on H or on H' , and hence maps to a hyperplane $W(C)$ that crosses U and crosses either H or H' .

Interpretation in U : Let P_0 be the image of P in U under the projection $N(U) \rightarrow U$, so that P_0 is the shortest path joining b_H to $b_{H'}$ in the tree U . Let $g \in U$ be the gate of the root b^* in P_0 , i.e. $g \in P_0$ is the unique point such that for all $p \in P_0$, any geodesic from p to b^* passes through g . Either g is contained in the interior of P_0 , or g is equal to one of the endpoints, so without loss of generality, suppose that $g \neq b_H$. This situation is shown in the center of Figure 7. We shall show that in fact every path in U from b^* to b_H must pass through $b_{H'}$ – that is to say, that $g = b_{H'}$ – and thus that $H' \prec H$.

A potential father of H' contacts $f(H)$: Let a'_0 be the 0-cube of P_0 adjacent to b_H , and let $a' \in P$ be the 0-cube mapping to a'_0 . Then there is a dual curve C in D emanating from the 1-cube aa' and ending on Q or on Q' . Let W be the hyperplane to which C maps. If C ends on Q , then W is a potential father of H . But $d(a'_0, b^*) < d(b_H, b^*)$ since a'_0 is closer than b_H to the gate g . This implies that $W = f(H)$. But all dual curves emanating from P map to distinct hyperplanes, a contradiction. Thus W is not a potential father of H , and hence $W \in PF(H')$. On the other hand, $W \perp f(H)$. It therefore remains to show that $H' \prec H$.

$b_{H'}$ is the gate: Every other dual curve C' emanating from P must end on Q' and thus map to a potential father of H' . Indeed, no such dual curve can cross C by minimality of the area of D . Hence every path in U from b^* to an interior vertex of P_0 must pass through $b_{H'}$, since V' is the father of H' , and thus $g = b_{H'}$. Hence each path from b^* to b_H passes through $b_{H'}$, and thus $H' \prec H$. \square

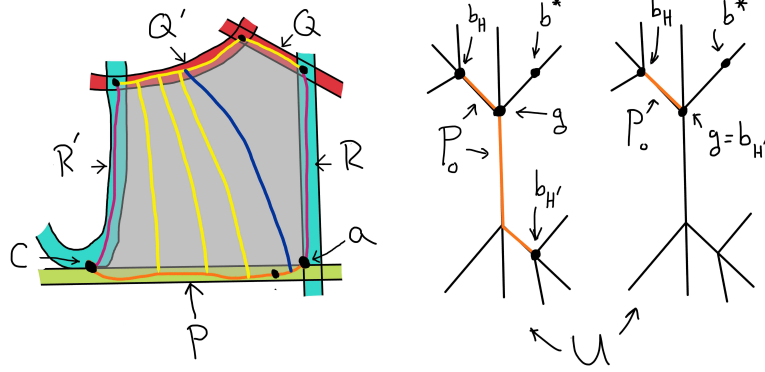


FIGURE 7. At left is the diagram D in the proof of Lemma 6. In the center is an a priori picture of the projection of P to U ; at right is the actual picture.

6. THE GRAPH $\Upsilon(U)$

Now, we define the following subgraph $\Upsilon(U)$ of $\Gamma(\mathbf{X})$: the vertices of $\Upsilon(U)$ are the hyperplanes of $\mathcal{R}(U)$ and two hyperplanes H and H' are adjacent in $\Upsilon(U)$ if and only if $H \perp H'$, the fathers $f(H)$ and $f(H')$ are different, and $f(H)$ and $f(H')$ do not contact. By Lemma 6, if H and H' are adjacent in $\Upsilon(U)$, then either $H \prec H'$ and the father $f(H')$ of H' contacts a potential father of H , or $H' \prec H$ and the father $f(H)$ of H contacts a potential father of H' . Note that $\Upsilon(U)$ is a subgraph of the grade-2 cluster \mathcal{C} centered at U . The graph $\Upsilon(U)$ can also be viewed as a subgraph of the intersection graph of iterated imprints of hyperplanes in $\mathcal{R}(U)$, by Lemma 5.

Our goal is to colour $\Upsilon(U)$. Since to colour the whole graph $\Upsilon(U)$ it is enough to colour each of its connected components, we will assume without loss of generality that $\Upsilon(U)$ is connected. To colour $\Upsilon(U)$, we will group the edges of $\Upsilon(U)$ into three spanning subgraphs $\Upsilon_0(U), \Upsilon_1(U), \Upsilon_2(U)$ of $\Upsilon(U)$ and colour each of these graphs separately.

Definition 8 (Root class, incoming neighbour, outgoing neighbour). The vertices of $\Upsilon(U)$ can be partitioned into subsets according to their roots: for each vertex b of U , let \mathcal{R}'_b be the set of hyperplanes $H \in \mathcal{R}(U)$ such that $b_H = b$. In other words, \mathcal{R}'_b is the set of hyperplanes H such that the iterated imprint $IJ(H, U)$ is rooted at b . The set \mathcal{R}'_b is the *root class* associated to the root b .

Let $H \in \mathcal{R}'_b$ and let $V = f(H)$ be its father. The hyperplane $H' \in \mathcal{R}(U)$ is an *incoming neighbour* of H if HH' is an edge of $\Upsilon(U)$ (and in particular $H \perp H'$), and the iterated

imprint $IJ(H', U)$ contains b (and in particular $H' \prec H$). More intuitively, H' is an incoming neighbour of H if HH' is an edge of $\Upsilon(U)$ and $H' \prec H$. Denote by $\mathcal{I}_b(H)$ the set of incoming neighbours of H .

By Lemma 6, for each $H' \in \mathcal{I}_b(H)$, we have that $f(H')$ contacts a potential father of H that crosses U . If H' is adjacent to H in $\Upsilon(U)$ and H' is not an incoming neighbour, then by Lemma 6, $H \prec H'$, and we call H' an *outgoing neighbour* of H .

The incoming neighbours of a fixed hyperplane are totally ordered by \prec ; while we do not make explicit use of this fact in colouring $\Gamma(\mathbf{X})$, it is a basic property of \prec .

Proposition 8 (Incoming neighbours). *For any vertex b of U and any $H \in R'_b$, the set $\mathcal{I}_b(H)$ of incoming neighbours of H is totally ordered by \prec and there is a hyperplane W such that W contacts $f(H)$, W is not a potential father of H , and W is a potential father of H_i for all $H_i \in \mathcal{I}_b(H)$.*

Proof. The roots $b_{H'}$ of all incoming neighbours $H' \in \mathcal{I}_b(H)$ of H are all different from the root b_H of H and all belong to the unique path of the tree U between b^* and b_H . Therefore the trace of the partial order \prec on $\mathcal{I}_b(H)$ is a total order $\{H_1, H_2, \dots, H_m\}$ of the incoming neighbours of H . For each i , let $V_i = f(H_i)$ and let $V = f(H)$; by definition, $H_i \prec H$.

Denote by P_0 the path in U from b^* to b , and let P be the path in $\mathcal{IF}(H, U)$ projecting to P_0 . Let a be the vertex of P mapping to b . Let $R \rightarrow N(V)$ be a shortest geodesic joining a to $N(V) \cap N(H)$.

For each i , let $P_i \rightarrow N(U)$ be a shortest path joining a to $N(V_i)$ and let \bar{P}_i be the image of P_i in U . Let $R_i \rightarrow N(V_i)$ be a shortest path joining the terminus a_i of P_i to $N(H_i)$, and let $Q_i, Q'_i \rightarrow N(H), N(H_i)$ be a pair of geodesics whose concatenation joins the terminus of R to the terminus of R_i . Let $D_i \rightarrow \mathbf{X}$ be a minimal area disc diagram for $P_i R_i (Q'_i)^{-1} Q_i^{-1} R^{-1}$, as in Figure 8.

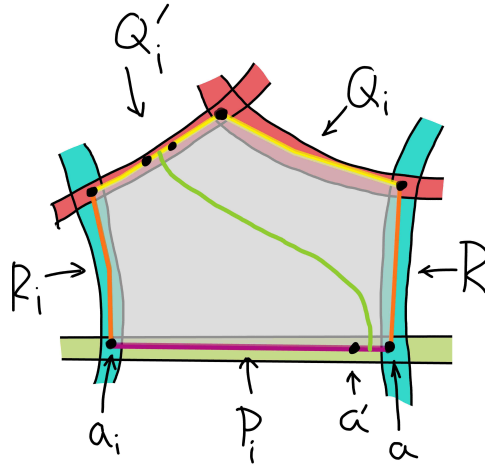


FIGURE 8. The diagram D_i .

Note that for all i , we have $\bar{P}_i \subseteq P_0$, since \bar{a}_i lies on the path from b to b^* since $H_i \prec H$. Hence we have that a_{i+1} lies on the path from a_i to b^* , and thus $\bar{P}_1 \subseteq \bar{P}_2 \subseteq \dots \subseteq \bar{P}_m \subseteq P_0$, i.e. $H_1 \prec H_2 \prec \dots \prec H_m \prec H$.

In particular, the initial 1-cube aa' of P_1 is contained in P_i for each i (P_1 contains at least one 1-cube since V and V_1 do not contact, by $\Upsilon(U)$ -adjacency of H and H_1). Hence, for all i , the dual curve C_i in D_i emanating from aa' maps to the same hyperplane W . Now C_i cannot end on P_i, R or R_i by minimality of the area, and thus C_i ends on Q_i or Q'_i . Moreover, C_i cannot end on Q_i . Indeed, if this were the case, then W would be a potential father of H . But since aa' projects to a 1-cube of P_0 , this would contradict the fact that V is the potential father whose imprint is closest to b^* . Thus C_i ends on Q'_i , and moreover C_i ends on a 1-cube of Q'_i that does not contain a 0-cube of Q_i . Hence W is a potential father of H_i for each i , and W crosses U and H_i . \square

6.1. Diagrams lying over edges in $\Upsilon(U)$. Let $H, H' \in \mathcal{R}(U)$ be hyperplanes such that $H'H$ is an edge of $\Upsilon(U)$ and $H' \prec H$, i.e. let H' be an incoming neighbour of H . Then, as in the proof of Lemma 6, there is a disc diagram $D \rightarrow \mathbf{X}$ associated to the edge $H'H$ as follows.

Let a', a be 0-cubes of $N(U)$ projecting to the roots b', b of $V' = f(H')$ and $V = f(H)$, respectively. Let $P \rightarrow N(U)$ be a geodesic segment joining a' to a , and let a' and a be chosen among the preimage points of b', b in such a way that $|P|$ is minimal. Let $R' \rightarrow N(V')$ and $R \rightarrow N(V)$ be geodesic segments respectively joining a' and a to closest 0-cubes of $N(H') \cap N(V')$ and $N(H) \cap N(V)$. Let $Q' \rightarrow N(H')$ and $Q \rightarrow N(H)$ be geodesic segments that have a single common 0-cube in $N(H) \cap N(H')$, so that the concatenation $Q'Q$ joins the terminal 0-cube of R' to the terminal 0-cube of R . Then there is a minimal-area disc diagram $D \rightarrow \mathbf{X}$ with boundary path $R'Q'QR^{-1}P^{-1}$; we say that D lies over the edge $H'H$, as shown in Figure 9.

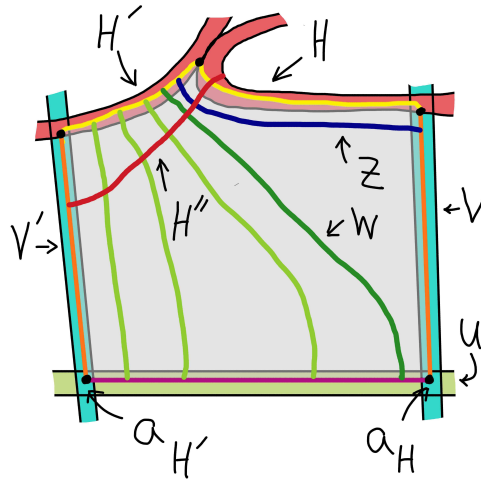


FIGURE 9. A diagram D that lies over the edge $H'H$ of $\Upsilon(U)$, with $H' \prec H$.

Analysis of a diagram D lying over $H'H$ reveals two hyperplanes, denoted $Z = Z(H'H)$ and $W = W(H'H)$ associated to the pair $H'H$, and the diagram D . Since H' and H have distinct, non-contacting fathers, we see that $|P| > 0$, and hence there is a dual curve L emanating from the terminal 1-cube of P (i.e. the 1-cube containing a) and mapping to a hyperplane W that crosses U and H' and contacts V , as in Lemma 6.

Now L cannot end on the terminal 1-cube of Q' (i.e. the 1-cube containing the 0-cube $Q' \cap Q$), since W cannot contact H . Hence there is a dual curve K emanating from the terminal 1-cube of Q' and ending on the terminal 1-cube of R . Indeed, K cannot end on Q' , on R' , or on Q for the usual reasons of minimal area, and K cannot end on P , for otherwise the hyperplane Z to which K maps would be a better choice of father for H than V . Thus K ends on R , and hence K , Q and the subtended part of R bound a triangular subdiagram, which must have area 0. In particular, K must end on the terminal 1-cube of R . Thus $Z \perp V$, $Z \perp H$, and $Z \perp H'$. This situation is depicted in Figure 9. Note that Z does not, in general, contact W since K and L may be separated by many dual curves traveling from Q' to R .

6.2. Separating osculators and the graph $\Upsilon_1(U)$. Denote by \mathbf{A} and \mathbf{B} the halfspaces associated to U . Recall that since we are colouring $\Upsilon(U)$, and to do so requires only that we colour each component, we have assumed that $\Upsilon(U)$ is connected. Therefore, all hyperplanes H of $\Upsilon(U)$ belong to one and the same halfspace defined by U , say to the halfspace \mathbf{A} . Let $\mathbf{A}(H)$ and $\mathbf{B}(H)$ be the complementary halfspaces associated to any hyperplane H belonging to \mathbf{A} , in particular to any hyperplane of $\Upsilon(U)$. Since U and H are not crossing, U belongs to one of these halfspaces, say in the halfspace $\mathbf{B}(H)$. Then $\mathbf{A}(H) \subset \mathbf{A}$ and $H \subset \mathbf{A}$ for any hyperplane $H \in \mathcal{R}(U)$.

Let $H \in \mathcal{R}(U)$. Then $d(H) \geq 1$, and thus there exists a hyperplane W such that $U \subset \mathbf{B}(W)$ and $H \subset \mathbf{A}(W)$, which is to say that W separates H from U . Since any two convex subspaces of \mathbf{X} are separated by finitely many hyperplanes, there exists a hyperplane $S(H)$ such that $S(H)$ osculates with H and separates H from U . Indeed, there must exist $S(H)$ separating H from U such that $S(H)$ is not separated from H by any hyperplane, and thus $S(H) \perp H$. This contact cannot be a crossing, for otherwise $S(H)$ would not separate H from U , as each of the intersections $\mathbf{A}(H) \cap \mathbf{B}(S(H))$, etc., would be nonempty. Accordingly, we define each hyperplane $S(H)$ that osculates with H and separates H from U to be a *separating osculator* of H .

Lemma 7. *Let $H \in \mathcal{R}(U)$. Then one of the following holds:*

- (1) *If $d(H) \geq 2$, then H has a unique separating osculator $S(H)$ and $f(H)$ crosses $S(H)$. Moreover, $S(H) \in \mathcal{R}(U)$.*
- (2) *If $d(H) = 1$, then H has at most two separating osculators, $S_1(H)$ and $S_2(H)$, and $S_1(H)$ and $S_2(H)$ either cross or coincide. Moreover, $S_1(H)$ and $S_2(H)$ are potential fathers of H , and $f(H)$ either crosses $S_i(H)$ or coincides with $S_i(H)$.*

Proof. If S_1 and S_2 separate H from U and osculate with H , then either $S_1 = S_2$ or S_1 and S_2 cross. Indeed, suppose that $H \subset \mathbf{A}(S_1) \cap \mathbf{A}(S_2)$ and $U \subset \mathbf{B}(S_1) \cap \mathbf{B}(S_2)$. Either

$\mathbf{A}(S_1) = \mathbf{A}(S_2)$, or (say) $\mathbf{A}(S_1) \subset \mathbf{A}(S_2)$ or $\mathbf{A}(S_1) \cap \mathbf{B}(S_2) \neq \emptyset$. In the first case, $S_1 = S_2$. In the second case, S_1 separates S_2 from H , a contradiction. In the third case, S_1 and S_2 must cross, since each of the quarter-spaces determined by S_1 and S_2 is nonempty.

Thus the set of separating osculators of H is a set of pairwise-crossing hyperplanes, and hence, since $\dim \mathbf{X} = 2$, there are at most two separating osculators, $S_1(H)$ and $S_2(H)$.

If $d(H) \geq 2$, then neither $S_1(H)$ nor $S_2(H)$ contacts U , and hence $f(H)$ is not a separating osculator of H . Since $N(f(H)) \cap N(H)$ and $N(f(H)) \cap N(U)$ are both nonempty, $f(H)$ contains points of $\mathbf{A}(S_i)$ and $\mathbf{B}(S_i)$ for $i \in \{1, 2\}$, i.e. $f(H)$ crosses $S_1(H)$ and $S_2(H)$. Since \mathbf{X} contains no pairwise-crossing triple of hyperplanes, we conclude that $S_1(H) = S_2(H)$, and denote by $S(H)$ the unique separating osculator of H . Moreover, since H is separated from U by at least two hyperplanes, $S(H)$ is separated from U by at least one hyperplane, and hence $d(S(H)) \geq 1$. On the other hand, since $f(H)$ crosses $S(H)$, we have $S(H) \in \mathcal{R}(U)$.

If $d(H) = 1$, then $S_1(H)$ and $S_2(H)$ osculate with U and are thus potential fathers of H . If F is a potential father of H that is distinct from $S_i(H)$, then F crosses both $S_1(H)$ and $S_2(H)$. Hence either $S_1(H) = S_2(H)$ or $S_1(H)$ and $S_2(H)$ are the unique potential fathers of H . \square

In summary, if $d(H) > 1$, then the unique separating osculator $S(H)$ crosses $f(H)$. If $d(H) = 1$, then since $S_1(H) = S_2(H)$ or they cross, we define $S(H)$ to be whichever of $S_1(H)$ or $S_2(H)$ has closer root in U to b^* . In this case, either $S(H) = f(H)$ or $S(H)$ crosses $f(H)$. If $f(H) = S(H)$, then H is *father-separated* as shown in Figure 10.

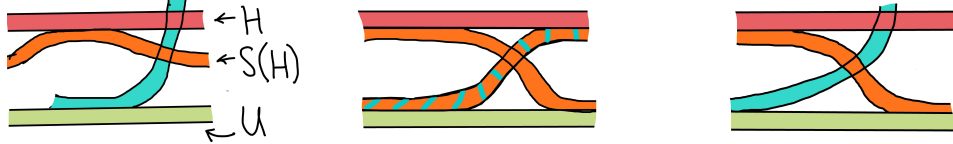


FIGURE 10. At left, $d(H) > 1$ and H has a unique separating osculator $S(H)$ that crosses $f(H)$. In the center, $d(H) = 1$ and H has two separating osculators, one of which is the father of H ; this is the father-separated case. At right, $d(H) = 1$, the father of H crosses H , and H has a unique separating osculator.

Define the graph $\Upsilon_1(U)$ as follows: $\Upsilon_1(U)$ has $\mathcal{R}(U)$ as the set of vertices and two hyperplanes H, H' are adjacent in $\Upsilon_1(U)$ if and only if H and H' are adjacent in $\Upsilon(U)$ and one of the following conditions holds: either $S(H) = S(H')$ or $S(H) = H'$ or $S(H') = H$.

Proposition 9. $\chi(\Upsilon_1(U)) \leq \Delta$.

Proof. We colour the hyperplanes of $\Upsilon_1(U)$ in the increasing hyperplane-distance $d(H)$ starting with the separating osculators contacting U (these hyperplanes do not belong to our graph $\Upsilon_1(U)$, but in order to make the colouring process uniform, we can suppose that they all received the same colour). Namely, suppose that W is the current hyperplane, W has been coloured and we want to colour all hyperplanes H such that $S(H) = W$. Notice that

the footprints in $N(W)$ of all such hyperplanes belong to one and the same bounding factor W' of $N(W)$ isomorphic to W and bounding the carrier $N(W)$ of W . Two hyperplanes H, H' with $S(H) = W = S(H')$ define an edge of $\Upsilon_1(U)$ if and only if H and H' contact. Since H and H' both osculate with W , by the Helly property we conclude that the footprints of H and H' in W' intersect. Therefore, in order to colour the hyperplanes having W as their separating osculator, it is enough to colour the intersection graph of the footprints of such hyperplanes in the tree W' so that all such hyperplanes receive a colour different from the colour of W . Any vertex v' of W' can belong to at most $\Delta - 1$ footprints of such hyperplanes (because v' has a neighbour v'' in the second hyperplane W'' bounding $N(W)$). Thus the intersection graph of the footprints on W' has clique number $\Delta - 1$. Since this graph is chordal and therefore perfect, we can colour it with $\Delta - 1$ colours. Taking into account the colour of W and extending this colouring process, we obtain a colouring of $\chi(\Upsilon_1(U))$ with at most Δ colours. \square

6.3. Father osculators and the graph $\Upsilon_2(U)$. Let $H'H$ be an edge of $\Upsilon(U)$ such that $H' \prec H$, and $H'H$ is not an edge of $\Upsilon_1(U)$, and $S(H')$ does not contact H . In other words, $H' \perp H$, the fathers of H and H' are distinct and do not contact, and the separating osculators of H and H' are distinct and distinct from H and H' .

Suppose also that H (respectively, H') does not separate U from H' (respectively, H), i.e. suppose that $U, H' \subset \mathbf{B}(H)$ and $U, H \subset \mathbf{B}(H')$ (this corresponds to the concurrency relation in event structures). Then we say that H' is a *father osculator* if the triplet of hyperplanes H', H , and $f(H)$ pairwise contact, and H' osculates with $f(H)$, and $S(H) \neq f(H)$. Let $\Upsilon_2(U)$ be the spanning subgraph of $\Upsilon(U)$ consisting of all edges $H'H$ of $\Upsilon(U)$ such that $H' \prec H$ and H' is a father osculator of H .

Proposition 10. $\chi(\Upsilon_2(U)) \leq \Delta$.

Proof. Let H be a hyperplane of $\mathcal{R}(U)$ having a father osculator. Let $S(H)$ be the separating osculator of H . Since $S(H) \neq f(H)$, the hyperplanes $S(H)$ and $f(H)$ cross, by Lemma 7. Moreover, since H osculates with $S(H)$ and contacts $f(H)$, we conclude that the intersection c_0 of the carriers of the hyperplanes $H, S(H)$, and $f(H)$ is a single 0-cube or a single 1-cube. Indeed, since \mathbf{X} is 2-dimensional, $N(f(H)) \cap N(S(H))$ is a single 2-cube c , each of whose 1-cubes is dual to $f(H)$ or $S(H)$, and thus $c \cap N(H) = c_0$ is a 0-cube if H and $f(H)$ osculate and a 1-cube dual to $f(H)$ if H and $f(H)$ cross. In the first case, set $c_0 := \{v_H\}$ and in the second case, set $c_0 := \{v_H, v'_H\}$, where v_H belongs to the halfspace (denote it by $\mathbf{A}(f(H))$) of $f(H)$ containing the root b^* of U (v'_H belongs to the complementary halfspace $\mathbf{B}(f(H))$).

Let H' be a father osculator of H . We claim that $v_H \in N(H')$. First we show that $c_0 \cap N(H') \neq \emptyset$. Since H' contacts H and $f(H)$, by the Helly property it suffices to show that H' and $S(H)$ contact. Since $H \subset \mathbf{A}(S(H))$ and $H' \perp H$, it suffices to show that $H' \cap \mathbf{B}(S(H)) \neq \emptyset$. Suppose not: then $H' \subset \mathbf{A}(S(H))$ and thus $S(H)$ separates H' from U . Since $H'H$ is not an edge of $\Upsilon_1(U)$, we conclude that $S(H') \neq S(H)$, whence $d(S(H)) < d(S(H'))$. Therefore $S(H')$ cannot separate H from U . Since $H' \subset \mathbf{A}(S(H'))$ and H' contacts H , necessarily $S(H)$ crosses H , contrary to the definition of edges of $\Upsilon_2(U)$. This shows that

the hyperplanes $H', H, f(H)$, and $S(H)$ pairwise contact, and therefore their carriers share a vertex of c_0 . Suppose that this vertex is v'_H and not v_H . Since H' osculates with $f(H)$, H' is contained in the halfspace $\mathbf{B}(f(H))$ of $f(H)$. But in this case, the root $b_{H'}$ of H' is also contained in $\mathbf{B}(f(H))$. Since b_H is contained in $\mathbf{A}(f(H))$, b_H lies on the unique path of U between $b_{H'}$ and b^* , and we obtain a contradiction with the assumption that $H'H$ is an edge of $\Upsilon(U)$ and $H' \prec H$. This contradiction shows that $v_H \in N(H')$.

Now, since v_H belongs to the carrier of any father osculator H' of H , H can have at most $\Delta - 1$ father osculators. Thus, in $\Upsilon_2(U)$ the incoming degree of any hyperplane H is at most $\Delta - 1$ and therefore $\Upsilon_2(U)$ can be coloured in Δ colours by the greedy algorithm following the orientation of edges defined by \prec . \square

6.4. The graph $\Upsilon_0(U)$. Let $\Upsilon_0(U)$ be the graph obtained by removing from $\Upsilon(U)$ all edges of the graphs $\Upsilon_1(U)$ and $\Upsilon_2(U)$, i.e., $H'H$ is an edge of $\Upsilon_0(U)$ if and only if $H'H$ is an edge of $\Upsilon(U)$ (i.e., $H' \prec H$, $H \perp H'$, and $f(H)$ and $f(H')$ are distinct and do not contact), $S(H)$ is different from H' and $S(H')$, and H' is not a father osculator of H . We will show that $\Upsilon_0(U)$ is bipartite and therefore can be coloured in two colours. Together with Propositions 9 and 10, this will show that the graph $\Upsilon(U)$ can be coloured in $2\Delta^2$ colours. We start with the following classification of edges of $\Upsilon(U)$.

Lemma 8. *Let $H'H$ be an edge of $\Upsilon(U)$. Then one of the following holds:*

- (1) $S(H)$ coincides with $S(H')$ or H' (and $H'H$ is an edge of $\Upsilon_1(U)$).
- (2) H' is a father osculator of H (and $H'H$ is an edge of $\Upsilon_2(U)$).
- (3) $S(H')$ crosses H .

Proof. Suppose that $H'H$ is not an edge of $\Upsilon_1(U)$. Then, by definition, we have that the hyperplanes $S(H), S(H'), H$, and H' are all distinct. We shall argue, using a disc diagram lying over the edge $H'H$, that if $S(H')$ fails to cross H , then $f(H), H$ and H' pairwise-osculte. Moreover, in the case that $S(H) = f(H)$, the same diagram shows that $S(H')$ crosses H .

Let $V' = f(H')$, $V = f(H)$, $S = S(H)$, and $S' = S(H')$. As in Section 6.1, let $P \rightarrow N(U)$ join the preimages of the roots of V' and V , let $R', R \rightarrow N(V'), N(V)$ be shortest geodesic segments joining the initial and terminal 0-cube of P , respectively, to $N(V') \cap N(H')$ and $N(V) \cap N(H)$, and let $Q'Q \rightarrow N(H') \cup N(H)$ be a shortest piecewise-geodesic segment joining the terminal 0-cubes of R' and R , so that the path $R'Q'QR^{-1}P^{-1}$ bounds a minimal-area disc diagram $D \rightarrow \mathbf{X}$ lying over the edge $H'H$, as in Figure 11.

Now consider the dual curve K' emanating from the terminal 1-cube of R' . Note that K' either ends on the initial 1-cube of Q or on some 1-cube of R , by minimality of the area of D . Moreover, K' maps to S' . Indeed, since S' separates H' from U , the geodesic segment R' must contain a 1-cube dual to S' . Let $R' = R''T$, where R'' is the subpath joining the initial 0-cube of R' to the initial 0-cube of the 1-cube c dual to S' , and T is the subpath, beginning with c and ending at the terminal 1-cube of R' . Let $s \in N(S') \cap N(H')$ be a 0-cube, which must exist since S' and H' osculate. Let $A \rightarrow N(H')$ join s to the terminal 0-cube of T and let $B \rightarrow N(S')$ join the initial 0-cube of T to s . Then BAT^{-1} is a closed path bounding

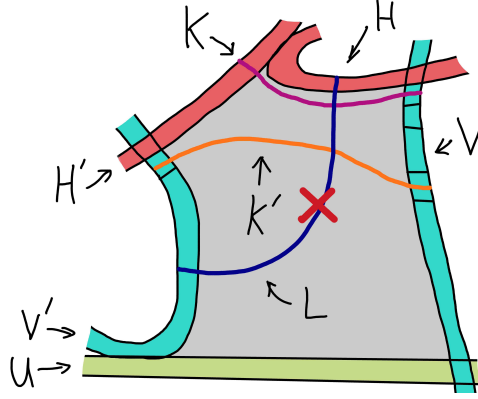


FIGURE 11. The diagram D lying over $H'H$ in the proof of Lemma 8.

a minimal-area disc diagram E . Any dual curve in E emanating from $T - c$ crosses A or B and thus leads to a trigon of pairwise-crossing dual curves; we conclude that $T = c$ and that S' crosses R' in its terminal 1-cube. Hence, since each 1-cube of \mathbf{X} is dual to a unique hyperplane, K' maps to S' .

If K' ends on Q , then S' crosses H , and we are done. Hence K' ends on R . If K' ends on the terminal 1-cube of R , then the above argument shows that K' maps to S and hence $S = S'$, and the proof is again complete.

The unique remaining possibility is that K' ends on R at some interior 1-cube, and the dual curve K emanating from the terminal 1-cube of R and mapping to S separates K' from Q , as shown in Figure 11. Note that S' and S both cross V , since K' and K end on 1-cubes of R . Hence S' cannot cross S , since otherwise S', S and V would be a pairwise-crossing triple of hyperplanes, contradicting 2-dimensionality of \mathbf{X} . Hence S' separates S from U , and therefore S is not a potential father of H , and in particular $S \neq V$.

Now suppose that $|Q| > 0$, so that there exists a dual curve L emanating from Q . L cannot end on Q' or on R , since that would lead to a trigon removal along the boundary path of D and a consequent area reduction. On the other hand, if L ends on P , then there would be a better choice of father for H , namely the hyperplane to which L maps, and hence L ends on Q' . But since K' travels from R' to R and emanates from the terminal 1-cube of R , the dual curves L and K' must cross, and map to distinct hyperplanes since R' is a geodesic segment. Hence S', W, V' pairwise-cross, where W is the hyperplane to which L maps, and this contradicts 2-dimensionality of \mathbf{X} .

Hence $|Q| = 0$ and, in particular, V contacts H' . On the other hand, V cannot cross H' . Indeed, since K ends on Q' , we see that S crosses H' and that V crosses S , and the absence of pairwise-crossing triples ensures that V and H' cannot cross. Thus V and H' osculate.

In summary, $S(H) \neq f(H)$ and $f(H), H$, and H' pairwise contact, and $f(H)$ osculates with H' . Hence H' is a father osculator, as shown in Figure 12.

□

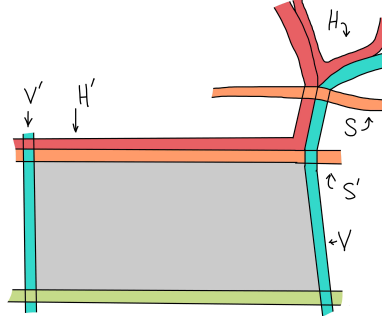


FIGURE 12. When the separating osculators of H' and H are distinct and distinct from H and H' , either H' is a father osculator of H or $S(H')$ crosses H .

The first step in proving that $\Upsilon_0(U)$ is bipartite is to show that it contains no triangles.

Lemma 9. *The graph $\Upsilon_0(U)$ is triangle-free.*

Proof. Let $C = (H_0, H_1, H_2)$ be a 3-cycle in $\Upsilon(U)$. Then without loss of generality, we have $H_1 \prec H_0 \prec H_2$. Indeed, one of the three vertices, say H_2 , does not precede either of the other two, and hence, by Lemma 6, we have $H_0 \prec H_2$ and $H_1 \prec H_2$. But by Lemma 6, since H_0 and H_1 are adjacent in $\Upsilon(U)$, they are comparable in the partial ordering \prec . Suppose, moreover, that the distance-sum $D(C) = d(H_0) + d(H_1) + d(H_2)$ is minimal among all 3-cycles in $\Upsilon(U)$. Denote by V_i the father of H_i , $i = 0, 1, 2$. By Lemma 8, $S(H_1)$ crosses H_0 and H_2 .

First suppose that $d(H_1) = 1$. If $S(H_1) = V_1$, then $b_{S(H_1)} \prec b_{V_0} \prec b_{V_2}$. But since $S(H_1)$ crosses H_0 and H_2 , by Lemma 8, we see that $S(H_1)$ is a potential father of H_0 and H_2 . Hence $S(H_1) = V_0 = V_2$ and we reach a contradiction of the fact that H_1H_0 and H_1H_2 are edges of $\Upsilon(U)$.

The remaining possibility is that in which $S(H_1)$ crosses (and is different from) V_1 and V_1 crosses at least one of U and H , by Lemma 7, as shown at right in Figure 10. If $S(H_1) = V_0$, then we reach a contradiction as above. Otherwise, since $S(H_1)$ crosses H_0 , and $S(H_1)$ is not the father of H_0 , and $H_1 \prec H_0$, the imprint of V_0 on U lies between the imprint of $S(H_1)$ and the imprint of V_1 , i.e. b_{V_0} lies on the unique path in U between b_{V_1} and $b_{S(H_1)}$. However, since $H_1, S(H_1), U$ pairwise-contact, the imprint of V_1 on U has nonempty intersection with the imprint of $S(H_1)$ on U , as illustrated in Figure 13. Since $b_{S(H_1)}$ is the closest point of the imprint of $S(H_1)$ to b_{V_1} , we see that $b_{S(H_1)}$ lies in the imprint of V_1 . But then b_{V_0} must lie in the imprint of V_1 , whence $V_0 \perp V_1$. This contradicts the fact that H_1H_0 is an edge of $\Upsilon(U)$. Hence $d(H_1) \geq 2$.

Now suppose that $d(H_1) \geq 2$. Then $S(H_1)$ is a hyperplane of $\mathcal{R}(U)$ by Lemma 7. The father of $S(H_1)$ is either V_1 or a hyperplane before V_1 . Therefore, $S(H_1) \prec H_0 \prec H_2$ holds, and $S(H_1)H_0$ and $S(H_1)H_2$ are both edges of $\Upsilon(U)$. Since $S(H_1)$ crosses H_0 and H_2 , neither of these edges is an edge of $\Upsilon_2(U)$. Now, suppose that $S(H_1)H_0$ is an edge of $\Upsilon_1(U)$. Since

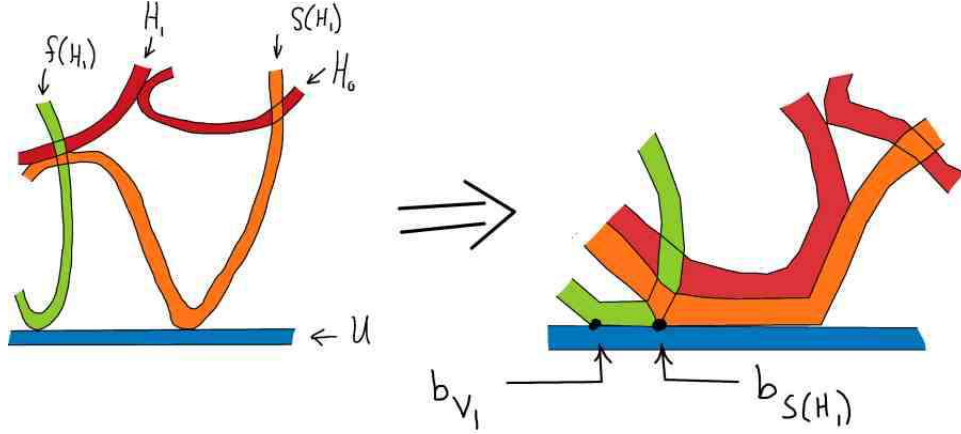


FIGURE 13. A heuristic picture showing that $b_{S(H_1)}$ lies in the imprint of V_1 .

$S(H_1)$ and H_0 cross, this is possible only if $S(S(H_1)) = S(H_0)$. But in this case, the disc diagram lying over the edge H_1H_0 will contain a trigon.

Indeed, suppose that $S(S(H_1)) = S(H_0)$ and let $D \rightarrow \mathbf{X}$ be a diagram lying over the edge H_1H_0 . Then the dual curve K emanating from the terminal 1-cube of R_1 and mapping to $S(H_1)$ ends on Q'_0 at the initial 1-cube. The subdiagram $D' \subset D$ bounded by $N(K)$, P , R_0 and the subtended part of Q'_0 lies over the edge $S(H_1)H_0$. The dual curve L emanating from the penultimate 1-cube of R_1 maps to $S(S(H_1))$, and by the assumption that $S(S(H_1)) = S(H_0)$, we have that L ends on the terminal 1-cube of R_0 . As usual, since V_1 does not contact V_0 , there is a dual curve M in D traveling from P to Q_1 , and M cannot end on the terminal 1-cube of Q_1 , for otherwise M would map to a hyperplane providing a better father for H_0 . Hence there must exist a dual curve N emanating from the terminal 1-cube of Q_1 and ending on R_0 . But N cannot end on the terminal 1-cube of R_0 , since that 1-cube is already the origin of L , and hence N must cross L . But the hyperplanes to which N and L map both cross V_0 , and thus cannot cross. Hence $S(S(H_1)) \neq S(H_0)$, as in Figure 14.

Hence $S(H_1)H_0$ and $S(H_1)H_2$ are not edges of $\Upsilon_1(U)$, showing that $C' = (H_0, S(H_1), H_2)$ is a 3-cycle of $\Upsilon_0(U)$. Since $D(C') < D(C)$, we obtain a contradiction with the minimality choice of C . \square

We now analyze cycles in $\Upsilon_0(U)$, with the goal of showing that there are no cycles of odd length. Let $C = (H_0, H_1, \dots, H_{n-1}, H_0)$ be a simple n -cycle in $\Upsilon_0(U)$. C is *induced* if for all $i \in \mathbb{Z}_n$ and $j \neq i \pm 1$, the hyperplanes H_i and H_j are not adjacent in $\Upsilon_0(U)$, i.e. either H_i and H_j do not contact, or their fathers contact or coincide, or they have a common separating osculator. Notice that if $\Upsilon_0(U)$ contains an odd cycle, then any its odd cycle of minimum length $n = 2k + 1$ is an induced cycle: indeed, if H_iH_j is an edge of $\Upsilon_0(U)$ that does not belong to C , then at least one of the paths of C connecting H_i, H_j has even length, and hence $\Upsilon_0(U)$ contains an odd-length cycle that is shorter than C .

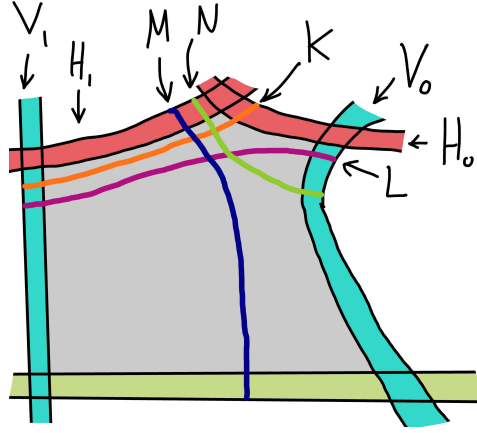


FIGURE 14. The diagram D lying over H_1H_0 contains a contradictory trigon when $S(S(H_1)) = S(H_0)$.

For a cycle $C = (H_0, H_1, \dots, H_{n-1}, H_0)$ of $\Upsilon_0(U)$, let $V_i = f(H_i)$ and let $b_i \in U$ be the root of the imprint of H_i on U , for each $i \in \mathbb{Z}_n$. Let \mathbf{A}_i and \mathbf{B}_i be the two halfspaces of \mathbf{X} defined by H_i so that $U \subset \mathbf{B}_i$. Cycles of $\Upsilon_0(U)$ have the following simple properties.

Lemma 10. *If $C = (H_0, H_1, \dots, H_n)$ is an induced path in $\Upsilon_0(U)$ and the hyperplane H_j is contained in the halfspace \mathbf{A}_i defined by the hyperplane H_i , then either $V_i = V_j$ or $H_i \prec H_j$.*

Proof. Since H_i separates H_j from U , the father V_j of H_j crosses H_i , and thus V_j is a potential father of H_i . From the definition of a father we conclude that either $V_i = V_j$ or $H_i \prec H_j$. \square

The hyperplane H_i of C is *normal* if exactly one of H_{i-1} and H_{i+1} is an incoming neighbour, and the other neighbour of H_i in C is outgoing. By Lemma 6, if H_i is not normal, then H_{i-1} and H_{i+1} are both incoming or both outgoing neighbours of H_i .

Lemma 11. *Any induced cycle C of $\Upsilon_0(U)$ of odd length $n = 2k + 1$ contains at least one normal hyperplane H_i .*

Proof. By Lemma 9, $n > 3$. Suppose that no hyperplane in C is normal. Then the hyperplanes of C can be partitioned into two sets \mathcal{I} and \mathcal{O} , where \mathcal{I} is the set of H_i for which both neighbours in C are incoming, and \mathcal{O} is the set of H_i for which both neighbours in C are outgoing. By definition, every edge of C joins an element of \mathcal{I} to an element of \mathcal{O} . Hence C is bipartite, and in particular has even length. \square

Lemma 12. *Any cycle of $\Upsilon_0(U)$ does not contain normal hyperplanes.*

Proof. We proceed by way of contradiction. Suppose that n is the smallest value for which there exists a cycle of length n containing a normal hyperplane. Now, among all minimal cycles of $\Upsilon(U)$ of length n containing normal hyperplanes, let $C = (H_0, H_1, \dots, H_{n-1}, H_0)$ has minimal distance sum $D(C)$. Let H_1 be a hyperplane of C having the closest root b_1 to b^* .

Then the neighbours H_0 and H_2 of H_1 in C are both outgoing neighbours of H_1 in $\Upsilon(U)$, and thus H_1 is not a normal hyperplane of C . We proceed as in the proof of Lemma 9. Let $S(H_1)$ be the separating osculator of H_1 . By Lemma 8, $S(H_1)$ crosses H_0 and H_2 . If $d(H_1) = 1$, then, as in the proof of Lemma 9, we reach the contradictory conclusion that the fathers of H_1 and H_0 coincide or contact.

Now suppose that $d(H_1) \geq 2$. Then $S(H_1)$ is a hyperplane of $\mathcal{R}(U)$ by Lemma 7. The father of $S(H_1)$ is either V_1 or a hyperplane before V_1 , by Lemma 10. Therefore, $S(H_1) \prec H_0 \prec H_2$ holds, and $S(H_1)H_0$ and $S(H_1)H_2$ are both edges of $\Upsilon(U)$. Since $S(H_1)$ crosses H_0 and H_2 , neither of these edges is an edge of $\Upsilon_2(U)$. Now, suppose that $S(H_1)H_0$ is an edge of $\Upsilon_1(U)$. Since $S(H_1)$ and H_0 cross, this is possible only if $S(S(H_1)) = S(H_0)$. But in this case, the disc diagram over the edge H_1H_0 will contain a trigon, as in the proof of Lemma 9. Hence $S(H_1)H_0$ and $S(H_1)H_2$ are not edges of $\Upsilon_1(U)$, and therefore they are edges of $\Upsilon_0(U)$, showing that $C' = (H_0, S(H_1), H_2, \dots, H_{n-1}, H_0)$ is a cycle of the graph $\Upsilon_0(U)$. Since $S(H_1) \prec H_1$, the choice of H_1 in C implies that if H_i is a normal hyperplane of C , then $i \neq 1$ and H_i is a normal hyperplane of C' . Since $D(C') < D(C)$, we obtain a contradiction with the minimality choice of C . This contradiction shows that no cycle of $\Upsilon_0(U)$ contains normal hyperplanes. \square

Proposition 11. *The graph $\Upsilon_0(U)$ is bipartite; therefore $\chi(\Upsilon_0(U)) = 2$.*

Proof. From Lemma 11 we know that any induced odd cycle of $\Upsilon_0(U)$ must contain a normal hyperplane. On the other hand, Lemma 12 asserts that no cycle of $\Upsilon_0(U)$ can contain a normal hyperplane. Since any graph containing odd cycles also contain induced odd cycles, we conclude that $\Upsilon_0(U)$ cannot contain any odd cycle, i.e. $\Upsilon_0(U)$ is bipartite. \square

Now, we are ready to prove the main result of this section:

Proposition 12. $\chi(\Upsilon(U)) \leq 2\Delta^2$.

Proof. To show that $\chi(\Upsilon(U)) \leq 2\Delta^2$, associate to each hyperplane H of $\Upsilon(U)$ the three colours of H in the colourings of the graphs $\Upsilon_0(U)$, $\Upsilon_1(U)$, and $\Upsilon_2(U)$ provided by Propositions 9, 10 and 11. Since $\chi(\Upsilon_0(U)) = 2$ and $\chi(\Upsilon_1(U)) \leq \Delta$, $\chi(\Upsilon_2(U)) \leq \Delta$, the hyperplanes of $\Upsilon(U)$ will be coloured with at most $2\Delta^2$ colours. Since each edge $H'H$ of $\Upsilon(U)$ is contained in at least one of the graphs $\Upsilon_0(U)$, $\Upsilon_1(U)$, and $\Upsilon_2(U)$, the triplets of H' and H differ in at least one coordinate; thus the resulting triplet-colouring is a correct colouring in at most $2\Delta^2$ colours of each connected component of $\Upsilon(U)$, and therefore of the whole graph $\Upsilon(U)$. \square

7. PROOF OF THEOREM 1

7.1. Colouring the contact graph $\Gamma(\mathbf{X})$. We now colour the contact graph $\Gamma(\mathbf{X})$, proving the first assertion of Theorem 1. The proof is divided into several steps.

Strategy: To show that $\chi(\Gamma(\mathbf{X}))$ is bounded by a function of the maximum degree Δ of $G(\mathbf{X})$, we take an arbitrary but fixed base hyperplane H_0 and partition the contact graph $\Gamma(\mathbf{X})$ into the spheres S_k , $k = 0, 1, \dots$, centered at H_0 . Now, if we will show that the subgraph of $\Gamma(\mathbf{X})$ induced by each sphere S_k can be coloured in $\alpha(\Delta)$ colours, then combining a

colouring of the spheres with even radius using the same set of $\alpha(\Delta)$ colours and a colouring of the spheres with odd radius using another set of $\alpha(\Delta)$ colours, we will obtain a correct colouring of $\Gamma(\mathbf{X})$ into $2\alpha(\Delta) = \epsilon(\Delta)$ colours.

In order to colour S_k in $\alpha(\Delta)$ colours, it suffices to colour each connected component (or each cluster) \mathcal{C} of S_k in the contact graph $\Gamma(\mathbf{X})$ with this number of colours. It follows from Corollary 3 that each such cluster \mathcal{C} has diameter at most 5. Moreover, it was shown in [Hag11] that \mathcal{C} has diameter at most 4. Therefore, if we pick an arbitrary hyperplane $V_0 := V_0^{\mathcal{C}}$ in \mathcal{C} , then all hyperplanes V of \mathcal{C} have distance at most 4 to V_0 , i.e., $\mathcal{C} \subset B_4(V_0)$, where $B_r(V_0) = \{V : \rho(V_0, V) \leq r\}$ is the ball of $\Gamma(\mathbf{X})$ of radius r centered at V_0 . Therefore, if we show that $B_5(V_0)$ or $B_4(V_0)$ can be coloured with $\alpha(\Delta)$ colours, then taking the restriction of this colouring to \mathcal{C} , we will obtain a colouring of \mathcal{C} into at most $\alpha(\Delta)$ colours. Repeating this colouring procedure for each cluster \mathcal{C} of S_k with the same set of $\alpha(\Delta)$ colours, we will obtain the required colouring of S_k .

Let $q(r)$ be the number of colours necessary to colour the ball $B_r(V_0)$ of radius r centered at V_0 . The main part of our proof is to establish the following recurrence $q(r) \leq q(r-1) \cdot q(r-2) \cdot (2\Delta) \cdot (2\Delta^2) + q(r-1)$, yielding a bound $q(4) \leq \alpha(\Delta)$. Suppose that the ball $B_{r-1}(V_0)$ has been coloured in $q(r-1)$ colours and let c be a colouring of $B_{r-1}(V_0)$ with this number of colours obtained in the recursive way. We will show how to extend c to a colouring of $B_r(V_0)$ using the required number of colours showing how to colour the hyperplanes from $S_r(V_0)$ using at most $q(r-1) \cdot q(r-2) \cdot (2\Delta) \cdot 2\Delta^2$ extra-colours.

Choosing fathers and grandfathers: Suppose that the hyperplanes of \mathbf{X} are graded according to their distance in $\Gamma(\mathbf{X})$ from H_0 . For each grade- r hyperplane H , with $r \geq 2$, fix once and for all a canonical path $\gamma(H)$ in $\Gamma(\mathbf{X})$ joining H_0 to H . This determines a grandfather $f^2(H) = \gamma(H)(r-2)$ for H , and a set of potential fathers V of H : as before, these are the hyperplanes V with $f^2(H) \perp V \perp H$. (The case $r < 2$ is dealt with separately below.) Now, fix a root vertex in each hyperplane. The choice of root in $f^2(H)$ determines a father $f(H)$ of H , as above.

Colouring: For each hyperplane V , consider a colouring c' in at most 2Δ colours (we use the same set of at most 2Δ colours for each hyperplane) of the families of footprints or imprints of the set of all hyperplanes H having V as their father (this colouring is provided by Proposition 6 showing that $\chi(\mathcal{F}(V)) \leq \chi(\mathcal{J}(V)) \leq 2\Delta$). Additionally, for each hyperplane U define a colouring c'' of the graph $\Upsilon(U)$ in at most $2\Delta^2$ colours (we use the same set of $2\Delta^2$ colours to colour the graph $\Upsilon(U)$ for each hyperplane U). This colouring is provided by the Proposition 12, which shows that $\chi(\Upsilon(U)) \leq 2\Delta^2$. Recall that $\Upsilon(U)$ has the set $\mathcal{R}(U)$ of all hyperplanes H with $f^2(H) = U$ as the set of vertices. For $H \in \mathcal{R}(U)$ let $c''(H)$ be the colour of H in the colouring of $\Upsilon(U)$ with at most $2\Delta^2$ colours. Notice that it suffices to define the colourings c' and c'' only on hyperplanes of grades $r-1$ and $r-2$, respectively.

Now, for a hyperplane H of grade r , we assign as a colour the ordered quadruplet

$$c(H) = (c(f(H)), c(f^2(H)), c'(F(H, f(H))), c''(H))$$

Clearly, the hyperplanes of $S_r(V_0)$ will be coloured with at most $q(r-1) \cdot q(r-2) \cdot (2\Delta) \cdot (2\Delta^2)$ colours. Notice also that two contacting grade r and grade $r-1$ hyperplanes will be coloured differently because we use new colours for colouring $S_r(V_0)$. It remains to show that c is a correct colouring of the hyperplanes of $S_r(V_0)$. This is the content of Lemma 13.

We conclude that $q(r) \leq 4 \cdot \Delta^3 \cdot q(r-1) \cdot q(r-2) + q(r-1)$. Notice that $q(0) = 1$ since $S_0 = B_0(V_0) = \{V\}$. On the other hand, $q(1) \leq 2\Delta$ because colouring the hyperplanes of the sphere S_1 is equivalent to colouring the intersection graph of their imprints in V_0 and this can be done with at most 2Δ colours by Proposition 6. Easy calculations show that $q(2) \leq 8\Delta^4 + 2\Delta$, that $q(3) \leq 64\Delta^8 + 16\Delta^5 + 8\Delta^4 + 2\Delta$, and, finally, for $\Delta \geq 2$, that

$$\begin{aligned} q(4) &\leq 2^{11}\Delta^{15} + 2^{10}\Delta^{12} + 2^8\Delta^{11} + 2^8\Delta^8 + 2^4\Delta^5 \\ &\leq 2^{11}(1 + 2^{-4} + 2^{-7} + 2^{-10} + 2^{-17})\Delta^{15} \\ &= \frac{140417}{2^6}\Delta^{15} = \alpha(\Delta). \end{aligned}$$

Hence each sphere S_k admits a colouring using at most $\alpha(\Delta)$ colours, and thus

$$\chi(\Gamma(\mathbf{X})) \leq 2\alpha(\Delta) = \epsilon(\Delta)$$

for $\Delta \geq 2$. This concludes the proof of the first assertion of Theorem 1, with $M \leq 4389$.

Correctness: The following lemma establishes that c is a correct colouring of the contact graph $\Gamma(\mathbf{X})$.

Lemma 13. *If $H, H' \in S_r(V_0)$ and $H \perp H'$, then $c(H) \neq c(H')$.*

Proof. By the weak combing property established in Corollary 2, the grandfathers of H and H' either contact or coincide. If $f^2(H) \perp f^2(H')$, then by induction $c(f^2(H)) \neq c(f^2(H'))$, whence $c(H) \neq c(H')$ because the quadruplets $c(H)$ and $c(H')$ differ in the second coordinate. So, further we will assume H and H' have the same grandfather, say $U = f^2(H) = f^2(H')$ (i.e., $H, H' \in \mathcal{R}(U)$.) Analogously, if H and H' have different but contacting fathers $f(H)$ and $f(H')$, then $c(f(H)) \neq c(f(H'))$, and thus $c(H) \neq c(H')$ because the quadruplets $c(H)$ and $c(H')$ differ in the first coordinate. On the other hand, if H and H' have the same father V , then Lemma 2 implies $F(H, V) \cap F(H', V) \neq \emptyset$, so that $c'(F(H, V)) \neq c'(F(H', V))$, whence $c(H) \neq c(H')$ because $c(H)$ and $c(H')$ differ in the third coordinate.

Finally, suppose that $H \perp H'$, $U = f^2(H) = f^2(H')$ and the fathers of H and H' are different and do not contact. By definition of the adjacency in the graph $\Upsilon(U)$, the hyperplanes H and H' are adjacent in $\Upsilon(U)$, and therefore the colours of H and H' are different in the colouring c'' of $\Upsilon(U)$. Hence $c(H) \neq c(H')$ because $c(H)$ and $c(H')$ differ in the fourth coordinate. \square

7.2. Embeddings in products of trees: colouring the crossing graph $\Gamma_{\#}(\mathbf{X})$. We now deduce from the existence of a finite colouring of $\Gamma(\mathbf{X})$ that \mathbf{X} isometrically embeds in the product of at most $M\Delta^{15}$ trees, using Proposition 5. Adopting the median graph point of view, one sees that Proposition 2 of [BCE10b] also implies that colouring $\Gamma(\mathbf{X})$ suffices to embed \mathbf{X} in the product of finitely many trees. Since $\Gamma_{\#}(\mathbf{X})$ is a subgraph of $\Gamma(\mathbf{X})$, we have

a colouring c of the vertices of $\Gamma_{\#}(\mathbf{X})$ by a set \mathcal{K} of at most $M\Delta^{15}$ colours. The result now follows from Corollary 1.

7.3. The nice labeling problem: colouring the pointed contact graph $\Gamma_{\alpha}(\mathbf{X})$. Let \mathbf{X}_{α} be a 2-dimensional CAT(0) cube complex, pointed at α , and suppose that 0-cubes in \mathbf{X} have maximal degree Δ and maximal out-degree Δ_0 . Let $\Gamma_{\alpha}(\mathbf{X})$ be the pointed contact graph. In view of first assertion of the theorem, it suffices to show that $\Delta \leq \Delta_0 + 2$ for any 2-dimensional CAT(0) cube complex \mathbf{X} . (In fact, $\Delta \leq \Delta_0 + n$ holds for any n -dimensional CAT(0) cube complex and the proof is a consequence of the fact that intervals in median graphs are distributive lattices [BH83].) Let α be the basepoint and suppose by way of contradiction that a vertex v of $G_{\alpha}(\mathbf{X})$ contains three incoming neighbours v_1, v_2, v_3 . From the definition of the basepoint order on $G(\mathbf{X})$ it follows that v_1, v_2, v_3 are closer to α than the vertex v , i.e., $v_1, v_2, v_3 \in I(\alpha, v)$. Denote by $u_{i,j}$ the median of the triplet α, v_i, v_j . Since v_i and v_j are at distance 2, $u_{i,j}$ is adjacent to v_i and v_j . The vertices $u_{1,2}, u_{1,3}$, and $u_{2,3}$ are pairwise distinct, otherwise $G(\mathbf{X})$ would contain a $K_{2,3}$, which is impossible in a median graph. Now, let u be the median of the triplet $u_{1,2}, u_{1,3}, u_{2,3}$. The vertex u is different from v and is adjacent to each vertex of this triplet. As a result, the vertices $v, v_1, v_2, v_3, u_{1,2}, u_{1,3}, u_{2,3}, u$ define a 3-dimensional cube of $G(\mathbf{X})$ contrary to the 2-dimensionality of \mathbf{X} .

8. PROOF OF THEOREM 2

In this section, we construct an example establishing Theorem 2 by applying to the construction in [Che11] the “recubulation” construction in [Hag11]. Given a CAT(0) cube complex \mathbf{X} , the contact graph $\Gamma(\mathbf{X})$ can be realized as the crossing graph of a larger cube complex $\mathfrak{R}(\mathbf{X})$ that contains \mathbf{X} as an isometrically embedded subcomplex; $\mathfrak{R}(\mathbf{X})$ is the *recubulation* of \mathbf{X} , whose construction is given in [Hag11]. In the next proposition, we review the construction of $\mathfrak{R}(\mathbf{X})$ and establish several useful properties.

Proposition 13. *Let \mathbf{X} be a CAT(0) cube complex and let Δ be the maximum degree of a 0-cube in \mathbf{X} (i.e. the cardinality of a largest clique in the contact graph $\Gamma(\mathbf{X})$). Then there exists a CAT(0) cube complex $\mathfrak{R}(\mathbf{X})$ and a combinatorial isometric embedding $\mathbf{X} \rightarrow \mathfrak{R}(\mathbf{X})$ such that:*

- (1) *The hyperplanes of $\mathfrak{R}(\mathbf{X})$ are in a bijection with those of \mathbf{X} .*
- (2) *$\Gamma(\mathbf{X}) = \Gamma_{\#}(\mathfrak{R}(\mathbf{X}))$.*
- (3) *$\dim \mathfrak{R}(\mathbf{X}) = \Delta$, and each 0-cube of $\mathfrak{R}(\mathbf{X})$ has degree at most $\Delta^2 + \Delta$.*

More generally, if $\Gamma_{\#}(\mathbf{X}) \subseteq \Gamma_{\alpha} \subseteq \Gamma(\mathbf{X})$, then there exists a CAT(0) cube complex \mathfrak{R}_{α} of dimension at most $\omega(\Gamma_{\alpha})$ and maximal degree at most $\Delta^2 + \Delta$ such that \mathfrak{R}_{α} contains an isometric copy of \mathbf{X} and the crossing graph of \mathfrak{R}_{α} is equal to Γ_{α} .

Proof. The idea of recubulation is to force each pair of osculating hyperplanes (corresponding to an osculation-edge in Γ_{α}) in \mathbf{X} to cross by the addition of a set of 2-cubes into which those hyperplanes extend; further cubes of higher dimension are added where necessary to satisfy the link condition of [Gro87] and make $\mathfrak{R}(\mathbf{X})$ a CAT(0) cube complex. The following

argument shows that, for any subgraph $\Gamma_\alpha \subseteq \Gamma(\mathbf{X})$ that contains $\Gamma_\#(\mathbf{X})$, there is a CAT(0) cube complex \mathfrak{R}_α with $\mathbf{X} \subseteq \mathfrak{R}_\alpha \subseteq \mathfrak{R}(\mathbf{X})$, such that \mathfrak{R}_α contains \mathbf{X} as an isometrically embedded subcomplex of dimension at most $\omega(\Gamma_\alpha)$ and degree at most $\Delta^2 + \Delta$.

The intermediate complex \mathbf{X}' : Let H and H' be osculating hyperplanes in \mathbf{X} . By definition, there exists a 0-cube v and distinct 1-cubes e, e' incident to v such that e is dual to H and e' to H' . Attach a 2-cube s to \mathbf{X} by identifying two consecutive 1-cubes of s with the e and e' respectively, so that $s \cap \mathbf{X}$ consists of the path ee' . Perform this procedure for each pair (e, e') of 1-cubes of \mathbf{X} corresponding to a pair of osculating hyperplanes. (If one is recubulating with respect to a subgraph $\Gamma_\alpha \subseteq \Gamma(\mathbf{X})$, then one only performs this construction on pairs (e, e') that realize an osculation-edge of Γ_α .) Denote the resulting cube complex by \mathbf{X}' .

Constructing $\mathfrak{R}(\mathbf{X})$ and \mathfrak{R}_α : By construction, each hyperplane $H \in \mathcal{H}$ of \mathbf{X} extends to a subspace $W(H)$ of \mathbf{X}' that separates $(\mathbf{X}')^0$ into exactly two components. Indeed, $W(H)$ consists of H , together with the midcube of s dual to the 1-cube e of $s \cap \mathbf{X}$ dual to H , for each new 2-cube s that was attached at the site of an osculation of H with some other hyperplane of \mathbf{X} . In the language of [HP98], $W(H)$ is a *wall* in $(\mathbf{X}')^0$ whose halfspaces $\mathfrak{h}(W(H))$ and $\mathfrak{h}^*(W(H))$ respectively contain the 0-skeleta of $\mathbf{A}(H)$ and $\mathbf{B}(H)$. More precisely, if H extends into a new 2-cube s of \mathbf{X}' , then H is dual in \mathbf{X} to a unique 1-cube e of s and $W(H)$ intersects s in a midcube c of s . $\mathfrak{h}(W(H))$ consists of $\mathbf{A}(H)$ together with the halfspace of each such s induced by c that contains the 0-cube of e lying in $\mathbf{A}(H)$.

By construction, the assignment $H \mapsto W(H)$ is bijective, and the walls $W(H)$ and $W(H')$ cross if and only if $H \not\perp H'$ (or, more generally, if and only if H and H' are adjacent in Γ_α). Let $\mathfrak{R}(\mathbf{X})$ be the cube complex dual to the wallspace $((\mathbf{X}')^0, \{W(H) : H \in \mathcal{H}\})$. By the definition of the cube complex dual to a wallspace (see e.g. [CN05]), the set of hyperplanes of $\mathfrak{R}(\mathbf{X})$ corresponds bijectively to the set of walls, and therefore to the set \mathcal{H} of hyperplanes of \mathbf{X} . This establishes assertion (1).

Let $v_0 \in \mathbf{X}^0 \subseteq (\mathbf{X}')^0$ be a 0-cube. Recall that a 0-cube $x \in \mathfrak{R}(\mathbf{X})$ is a choice $x(W(H))$ of halfspace associated to each wall $W(H)$ such that $v_0 \in x(W(H))$ for all but finitely many hyperplanes H of \mathbf{X} and $x(W(H)) \cap x(W(H')) \neq \emptyset$ for all H, H' . In particular, if $x \in \mathbf{X}$ is a 0-cube, then one can make a choice $\phi(x)(W(H))$ of halfspace of $(\mathbf{X}')^0$, for each wall $W(H)$, by declaring $\phi(x)(W(H))$ to be the halfspace of \mathbf{X}' containing v_0 if and only if $x(H)$ contains v_0 . This yields an injective map $x \mapsto \phi(x)$ from \mathbf{X}^0 to $\mathfrak{R}(\mathbf{X})^0$. Moreover, it is easily checked that $\phi(x)$ and $\phi(x')$ differ on the wall $W(H)$ if and only if x and x' are separated in \mathbf{X} by the hyperplane H . It follows that \mathbf{X} embeds in $\mathfrak{R}(\mathbf{X})$ in such a way that each hyperplane H of \mathbf{X} , viewed as a subcomplex of $\mathfrak{R}(\mathbf{X})$, is equal to the intersection of \mathbf{X} with a hyperplane \widehat{H} of $\mathfrak{R}(\mathbf{X})$.

Isometric embedding: By the construction of $\mathfrak{R}(\mathbf{X})$, any hyperplane H of \mathbf{X} is the trace on \mathbf{X} of a hyperplane of $\mathfrak{R}(\mathbf{X})$ and, vice-versa, each hyperplane of $\mathfrak{R}(\mathbf{X})$ is the extension to $\mathfrak{R}(\mathbf{X})$ of a hyperplane of \mathbf{X} . Therefore any two 0-cubes x, y of \mathbf{X} are separated in $\mathfrak{R}(\mathbf{X})$ and \mathbf{X} by the same number of hyperplanes. Hence, the graph $G(\mathbf{X})$ is isometrically embedded in the 1-skeleton $G(\mathfrak{R}(\mathbf{X}))$ of $\mathfrak{R}(\mathbf{X})$.

Comparing $\Gamma(\mathbf{X})$ and $\Gamma_{\#}(\mathfrak{R}(\mathbf{X}))$: If $H \perp H'$, then the walls $W(H)$ and $W(H')$ cross, hence $\Gamma(\mathbf{X}) \subseteq \Gamma_{\#}(\mathfrak{R}(\mathbf{X}))$. Conversely, if the walls $W(H)$ and $W(H')$ cross, then either H and H' already cross in \mathbf{X} , or $W(H)$ and $W(H')$ cross in a 2-cube s of \mathbf{X}' with the property that $s \cap \mathbf{X}$ is a path ee' with e dual to H and e' to H' . Hence $H \perp H'$, whence $\Gamma_{\#}(\mathfrak{R}(\mathbf{X})) \subseteq \Gamma(\mathbf{X})$, establishing thus assertion (2).

Bounds on the dimension: Let $\Gamma_{\alpha} \subseteq \Gamma(\mathbf{X})$ be a subgraph containing $\Gamma_{\#}(\mathbf{X})$. Let \mathfrak{R}_{α} be the recubulation of \mathbf{X} corresponding to Γ_{α} and let $\mathfrak{R}(\mathbf{X})$ be that corresponding to $\Gamma(\mathbf{X})$, i.e. hyperplanes in \mathfrak{R}_{α} cross if and only if the corresponding hyperplanes of \mathbf{X} are adjacent in Γ_{α} , and hyperplanes in $\mathfrak{R}(\mathbf{X})$ cross if and only if they contact in \mathbf{X} . By construction, $\mathbf{X} \subseteq \mathfrak{R}_{\alpha} \subseteq \mathfrak{R}(\mathbf{X})$, so that it suffices to bound the dimension and degree of $\mathfrak{R}(\mathbf{X})$. Since a maximal family of pairwise-crossing hyperplanes in $\mathfrak{R}(\mathbf{X})$ corresponds bijectively to a maximal family of pairwise-contacting hyperplanes in \mathbf{X} , it is clear that $\dim(\mathfrak{R}(\mathbf{X})) \leq \Delta$, proving the first inequality in assertion (3). (In fact, an identical argument shows that \mathfrak{R}_{α} has dimension bounded by the clique number $\omega(\Gamma_{\alpha})$ of Γ_{α} .)

The intersection with \mathbf{X} of a maximal cube of $\mathfrak{R}(\mathbf{X})$: For each maximal cube C of $\mathfrak{R}(\mathbf{X})$, we shall show that $C^* = C \cap \mathbf{X}$ is nonempty and C^* contains a 1-cube dual to each hyperplane of $\mathfrak{R}(\mathbf{X})$ that crosses C . Indeed, let $0 < d \leq \Delta$ be the dimension of C and let $\hat{H}_1, \dots, \hat{H}_d$ be the hyperplanes of $\mathfrak{R}(\mathbf{X})$ that cross C . For $1 \leq i \leq d$, let $H_i = \hat{H}_i \cap \mathbf{X}$ be the corresponding hyperplane of \mathbf{X} . Now, $K = \bigcap_{i=1}^d N(H_i) \neq \emptyset$. Indeed, since the hyperplanes \hat{H}_i pairwise-cross, the hyperplanes H_i pairwise-contact in \mathbf{X} , whence K is a nonempty convex subcomplex of \mathbf{X} by the Helly property. Suppose that there exists a hyperplane \hat{H} of $\mathfrak{R}(\mathbf{X})$ that separates K from C . Any hyperplane \hat{H}_i that crosses both K and C must cross \hat{H} , and thus $\{\hat{H}_i\}_{i=1}^d \cup \{\hat{H}\}$ is a family of pairwise-crossing hyperplanes in $\mathfrak{R}(\mathbf{X})$, contradicting the fact that C is a maximal cube. Hence no hyperplane of $\mathfrak{R}(\mathbf{X})$ can separate K from C .

Claim 1: $K \subseteq C$.

Proof of Claim 1: Suppose by way of contradiction that there exists a 0-cube $k \in K - C$. Then some hyperplane of $\mathfrak{R}(\mathbf{X})$ separates k from C . Among the hyperplanes of $\mathfrak{R}(\mathbf{X})$ separating k from C , let \hat{H} be a closest one to k . Then $k \in N(\hat{H})$. Let kk' be the 1-cube of $\mathfrak{R}(\mathbf{X})$ dual to \hat{H} . We assert that \hat{H} crosses any hyperplane $\hat{H}_i, i = 1, \dots, d$, which crosses the cube C . Let $u_i v_i$ be a 1-cube of C dual to \hat{H}_i . Since $k \in K \subseteq N(H_i)$, there exists a 1-cube kk_i of \mathbf{X} dual to H_i and therefore to \hat{H}_i . Suppose that k and u_i belong to the same halfspace, say $\mathbf{A}(\hat{H}_i)$, of $\mathfrak{R}(\mathbf{X})$ defined by \hat{H}_i , while k_i and v_i belong to the complementary halfspace $\mathbf{B}(\hat{H}_i)$.

The vertex-set C^0 of the cube C is a convex subset of the median graph $G(\mathfrak{R}(\mathbf{X}))$ and, since the convex sets of median graphs are gated, C^0 is a gated subset of $G(\mathfrak{R}(\mathbf{X}))$. Let x be the gate of k in C^0 . From the choice of \hat{H} , we conclude that k' belongs in $G(\mathfrak{R}(\mathbf{X}))$ to the interval $I(k, x)$. Since, $x \in I(k, u)$ for any $u \in C$, in particular $x \in I(k, v_i)$, necessarily $k' \in I(k, v_i)$. Analogously, since $k \in \mathbf{A}(\hat{H}_i)$ is adjacent to k_i and $k_i, v_i \in \mathbf{B}(\hat{H}_i)$, k_i lies on a shortest path between k and v_i , thus $k_i \in I(k, v_i)$. Let v be the median in $G(\mathfrak{R}(\mathbf{X}))$ of the triplet v_i, k_i, k' (recall that $\{v\} = I(v_i, k_i) \cap I(k_i, k') \cap I(k', v_i)$). Then $v \neq k$ and vk', vk_i are

1-cubes of $\mathfrak{R}(\mathbf{X})$. The 0-cubes v, k', k, k_i define a 4-cycle of $G(\mathfrak{R}(\mathbf{X}))$ and therefore a 2-cube of $\mathfrak{R}(\mathbf{X})$. This implies that the 1-cube vk' is dual to the hyperplane \widehat{H} while the 1-cube vk_i is dual to the hyperplane \widehat{H}_i , hence \widehat{H} and \widehat{H}_i cross in $\mathfrak{R}(\mathbf{X})$. As a result, we conclude that $\{\widehat{H}_i\}_{i=1}^d \cup \{\widehat{H}\}$ is a family of pairwise-crossing hyperplanes of $\mathfrak{R}(\mathbf{X})$, contradicting the fact that C is a maximal cube of $\mathfrak{R}(\mathbf{X})$. This contradiction shows that indeed $K \subseteq C$, thus proving that C^* is nonempty. \square

Claim 2: C^* contains a 1-cube dual to each hyperplane of $\mathfrak{R}(\mathbf{X})$ that crosses C .

Proof of Claim 2: Consider any hyperplane \widehat{H}_i crossing the cube C . If \widehat{H}_i crosses K , then necessarily \widehat{H}_i crosses C^* and we are done. So, suppose that \widehat{H}_i is disjoint from K . Since K is contained in the carrier of $H_i = \widehat{H}_i \cap \mathbf{X}$, any 0-cube k of K belongs to a 1-cube kk' of \mathbf{X} dual to H_i . On the other hand, since $k \in K \subseteq C$ and \widehat{H}_i crosses the cube C , necessarily there exists a 1-cube kk'' of C dual to \widehat{H}_i . Since the 1-cube kk' is also dual to \widehat{H}_i , we conclude that $k' = k''$, i.e., $k' \in C \cap \mathbf{X} = C^*$, whence kk' is a 1-cube of C^* dual to \widehat{H}_i . \square

Bounds on the maximum degree: Next we will show that the degree of any 0-cube x of $\mathfrak{R}(\mathbf{X})$ is bounded by $\Delta^2 + \Delta$. First suppose that $x \in \mathbf{X}$. There are at most Δ 0-cubes in \mathbf{X} adjacent to x . If $y \in \mathfrak{R}(\mathbf{X}) - \mathbf{X}$ is a 0-cube adjacent to x that was added to \mathbf{X} during recubulation, then by construction, there is a path $[x, y, z]$ of length 2 in $\mathfrak{R}(\mathbf{X})$ such that $z \in \mathbf{X}^0$ and $P = [x, y, z]$ is a concatenation of two 1-cubes lying on the boundary of a 2-cube $s \subset \mathfrak{R}(\mathbf{X})$ that was added during recubulation. Let $Q = [x, w, z]$ be another path of length 2 of s . For each $y \in \mathfrak{R}(\mathbf{X}) - \mathbf{X}$ adjacent to x , there is thus a 0-cube $z = z(y) \in \mathbf{X}$ at distance 2 from x such that, for some $w(y) \in \mathbf{X}$ adjacent to x , the 4-cycle $[x, w(y), z(y), y, x]$ bounds a 2-cube in $\mathfrak{R}(\mathbf{X})$ that does not appear in \mathbf{X} . Moreover, each path $[x, y', z(y)]$ with $y' \in \mathfrak{R}(\mathbf{X}) - \mathbf{X}$ lies on the boundary of a 2-cube s' with the same dual hyperplanes as s , so since the hyperplanes dual to 1-cubes incident to x are all distinct, $y' = y$. Hence the assignment $y \mapsto z(y)$ is injective, and the degree of x is thus bounded by the number of 0-cubes of \mathbf{X} at distance 1 or 2 from x , i.e. by $\Delta^2 + \Delta$.

Now suppose that $x \in \mathfrak{R}(\mathbf{X}) - \mathbf{X}$ and let \mathcal{C} be the set of all maximal cubes of $\mathfrak{R}(\mathbf{X})$ containing x .

Claim 3: For each C in \mathcal{C} , the intersection C^* of C and \mathbf{X} is non-empty and convex in \mathbf{X} .

Proof of Claim 3: It was shown in Claim 1 that $C^* \neq \emptyset$. Now, since each cube C of $\mathfrak{R}(\mathbf{X})$ is convex (namely, the set of 0-cubes of C is convex in the graph $G(\mathfrak{R}(\mathbf{X}))$) and $G(\mathbf{X})$ is an isometric subgraph of $G(\mathfrak{R}(\mathbf{X}))$, necessarily the set of vertices of C^* is convex in $G(\mathbf{X})$, hence C^* is convex in \mathbf{X} . \square

Claim 4: For all C_1, C_2 in \mathcal{C} , the intersection of C_1^* and C_2^* is nonempty.

Proof of Claim 4: Suppose by way of contradiction that $C_1^* \cap C_2^* = \emptyset$. Since by Claim 3 C_1^* and C_2^* are convex in \mathbf{X} and median graphs satisfy the Kakutani separation property (see [vdV93], Chapter I.3), there exists a hyperplane H of \mathbf{X} separating C_1^* from C_2^* . This hyperplane cannot cross either of the subcomplexes C_1^* or C_2^* . On the other hand, since C_1

and C_2 both contain x , the hyperplane H extends to a hyperplane of $\mathfrak{R}(\mathbf{X})$ that crosses C_1 or C_2 and, as proved in Claim 2, H is dual to a 1-cube of C_1^* or C_2^* . This is a contradiction. \square

From Claims 3 and 4, and the Helly property, it follows that the intersection of all of the C^* is nonempty as C varies in \mathcal{C} , since \mathcal{C} is finite. In other words, the intersection of all maximal cubes of $\mathfrak{R}(\mathbf{X})$ that contain x contains a 0-cube y of \mathbf{X} . Hence the degree in $\mathfrak{R}(\mathbf{X})$ of x is bounded by the degree of y in $\mathfrak{R}(\mathbf{X})$, which was shown before to be at most $D^2 + D$. This complete the proof of the assertion (3) of Proposition 13. \square

From Propositions 13 and 5, we now obtain:

Corollary 4. *For any CAT(0) cube complex \mathbf{X} , if there is an isometric embedding $\mathfrak{R}(\mathbf{X}) \rightarrow \mathbf{Y}$, where \mathbf{Y} is the product of n trees, then $n \geq \chi(\Gamma(\mathbf{X}))$. In particular, if there exists a CAT(0) cube complex \mathbf{X} with maximum degree Δ such that the chromatic number of $\Gamma(\mathbf{X})$ is infinite, then there exists a CAT(0) cube complex, namely $\mathfrak{R}(\mathbf{X})$, such that the maximum degree of a 0-cube in $\mathfrak{R}(\mathbf{X})$ is at most $\Delta^2 + \Delta$, and $\mathfrak{R}(\mathbf{X})$ does not embed isometrically in the product of finitely many trees.*

The main ingredient in the remaining part of the proof of Theorem 2 is the following result from [Che11].

Proposition 14. *There exists $\Delta < \infty$ such that, for each $n \geq 0$, there exists a finite, 4-dimensional pointed CAT(0) cube complex \mathbf{X}'_n such that each 0-cube of \mathbf{X}'_n has degree at most Δ and the contact graph of \mathbf{X}'_n contains a subgraph $\Gamma_{n,\alpha}$ such that $\Gamma_{\#}(\mathbf{X}'_n) \subseteq \Gamma_{n,\alpha}$ and $\Gamma_{n,\alpha}$ has clique number at most 5 and chromatic number greater than n .*

Remark 6. The construction in [Che11] shows that we can take $\Delta = 8$. The graph $\Gamma_{n,\alpha}$ is the pointed contact graph of \mathbf{X}'_n pointed at a particular 0-cube α_n .

The construction of \mathbf{X}'_n relies on an example due to Burling [Bur65]. A rigorous proof of Proposition 14, using Burling's construction, is given in [Che11]; here we summarize the basic notions and provide a sketch of the proof.

A (3-dimensional) *box* is a closed parallelepiped in \mathbb{R}^3 whose edges are parallel to the coordinate axes. Given a (finite) collection \mathcal{B} of boxes, let $\Omega(\mathcal{B})$ denote the intersection graph of \mathcal{B} , and define $\omega(\mathcal{B})$ to be the clique number of \mathcal{B} and $\chi(\mathcal{B})$ to be the chromatic number of $\Omega(\mathcal{B})$. As described in the survey of Gyárfás [GLB03], Burling constructed for each n a collection \mathcal{B}_n of boxes such that $\omega(\mathcal{B}_n) = 2$ and $\chi(\mathcal{B}_n) > n$ for all $n \geq 0$.

Following [Che11], Construction 1 takes as its input a suitably-defined collection \mathcal{B} of axis-parallel boxes and returns a 4-dimensional pointed CAT(0) cube complex \mathbf{X}'' whose pointed contact graph has clique number at most 5 and chromatic number greater than n .

Construction 1. We first define a certain cube complex K arising from a box and an associated family of planes in \mathbb{R}^3 , then produce from K a CAT(0) cube complex \tilde{K} of dimension 4 whose pointed contact graph has the desired colouring properties. We then apply this construction in the context of Burling's example to build the cube complexes \mathbf{X}'_n .

The box complex: Let B_0 be a box with one corner at the origin in \mathbb{R}^3 . Any family of hyperplanes in \mathbb{R}^3 (i.e. 2-dimensional, axis-parallel affine subspaces) that cross the interior of B_0 partitions B_0 into a family of boxes and thus defines a “box complex” realized by B_0 . A cell of B_0 is an “elementary box”. By rescaling each of the constituent elementary boxes of B_0 so that the sides have length 1, we obtain a CAT(0) cube complex K , defined by B_0 and the initial family of hyperplanes. Note that K decomposes as a product, but is not the cube complex dual to the wallspace whose underlying set is B_0 and whose walls are the initial 2-dimensional hyperplanes. Indeed, each maximal family of m parallel 2-planes determines $m + 1$ walls. Instead, these 2-planes are part of the 2-skeleton of K , and each is parallel to a hyperplane. Note that K decomposes as a product of at most three 1-dimensional CAT(0) cube complexes and hence has dimension at most 3. The dimension of K is equal to the number of coordinate planes through the origin in \mathbb{R}^3 that are parallel to at least one of the given 2-planes.

Lifting to produce the pointed cube complex \tilde{K} : Let \mathcal{B} be a finite family of boxes. Then there exists a box B_0 whose interior contains each $B \in \mathcal{B}$, and the family \mathcal{B} induces a CAT(0) cubical structure on B_0 . Indeed, each of the twelve 2-planes determining $B \in \mathcal{B}$ becomes part of the cubical structure on B_0 constructed above. For each $B_i \in \mathcal{B}$, let K^i be the cubical subcomplex of K consisting of the cubes arising as cells of B_0 lying in B_i . Then K^i is a product of (at most) three subdivided intervals and is a convex subcomplex of K .

\tilde{K} is defined as a subset of \mathbb{R}^{m+3} , where $m = |\mathcal{B}|$. First, let B_0 lie in the subspace $\mathbb{R}^3 \subset \mathbb{R}^{m+3}$ consisting of points of the form $(0, 0, \dots, a, b, c)$ with $a, b, c \geq 0$. Then for each $B_i \in \mathcal{B}$, define $0 \leq a'_i < a''_i$, $0 \leq b'_i < b''_i$, $0 \leq c'_i < c''_i$ be numbers such that B_i consists of points of the form $p = (0, 0, \dots, a, b, c)$ with $a'_i \leq a \leq a''_i$, $b'_i \leq b \leq b''_i$, $c'_i \leq c \leq c''_i$. For $1 \leq i \leq m$, let s_i be the unit segment of the i^{th} coordinate axis of \mathbb{R}^{m+3} and let $\tilde{B}_i = s_i \times B_i$, so that \tilde{B}_i consists of points of the form $p = (p_1, p_2, \dots, p_m, a, b, c)$, where $(a, b, c) \in [a'_i, a''_i] \times [b'_i, b''_i] \times [c'_i, c''_i]$ and $p_j = \delta_{ij} \cdot [0, 1]$. This gives rise to a box hypergraph $\tilde{\mathcal{B}}$ in \mathbb{R}^{m+3} consisting of the boxes \tilde{B}_i with $1 \leq i \leq m$.

Note that each elementary box C of K gives rise to a 4-cube \tilde{C}_i in \mathbb{R}^{m+3} , isomorphic to $C \times s_i$, for each 3-dimensional box B_i in K that contains C . \tilde{C}_i is a *lifted elementary box*. Let \tilde{K}' be the 4-dimensional box complex consisting of all lifted elementary boxes \tilde{C}_i , together with all of the elementary boxes corresponding to 3-cubes of K . Let \tilde{K} be the cube complex obtained by rescaling the edges of each elementary box so that they have length 1. Note that $K \subset \tilde{K}$.

In Lemma 1 of [Che11], it is shown that the 1-skeleton of \tilde{K} is a median graph, and therefore that \tilde{K} is a CAT(0) cube complex, by taking gated amalgams of the 1-skeleta of the complexes K^i , which are themselves median graphs.

We now let α be the corner of B_0 corresponding to the origin in $\mathbb{R}^3 \subset \mathbb{R}^{m+3}$ and point \tilde{K} from α , i.e. orient each 1-cube of \tilde{K} away from the halfspace of its dual hyperplane that contains α . Let Γ_α be the corresponding pointed contact graph.

The pointed contact graph Γ_α : Lemma 4 of [Che11] states that, if $\omega = \omega(\mathcal{B})$ is the clique number of the intersection graph of the family \mathcal{B} , then the clique number of the

contact graph $\Gamma(\tilde{K})$ (i.e. the maximum cardinality of a 0-cube of \tilde{K}) is at most $\omega + 6$, and the maximum number of 1-cubes oriented outward from a 0-cube of \tilde{K} , i.e. the clique number of Γ_α , is exactly $\omega + 3$. On the other hand, since Γ_α contains the intersection graph of \mathcal{B} , we have that $\chi(\Gamma_\alpha) \geq \chi(\mathcal{B})$.

Application to Burling's boxes: For each $n > 0$, let \mathcal{B}_n be a finite family of boxes such that $\omega(\mathcal{B}_n) = 2$ and $\chi(\mathcal{B}_n) > n$. Then since \mathcal{B}_n is finite, there exists a box $B_{0,n}$ that contains each box $B \in \mathcal{B}_n$. For each box $B \in \mathcal{B}_n$, there are 12 2-planes crossing $B_{0,n}$ that together pass through the eight corners of B and determine B as a box in \mathbb{R}^3 . The set of all such 2-planes determines a CAT(0) cube complex from $B_{0,n}$ as above, denoted K_n . Let \mathbf{X}'_n be the 4-dimensional CAT(0) cube complex constructed from K_n as above. Then \mathbf{X}'_n has maximal degree $\Delta = \omega(\mathcal{B}_n) + 6 = 8$.

Moreover, let α_n and β_n be a pair of opposite corners of $B_{0,n}$. Then α_n and β_n are 0-cubes of K_n and lift to distinct 0-cubes of \mathbf{X}'_n ; denote these lifts also by α_n and β_n . Letting \mathbf{X}'_n be pointed from α_n , we see that the pointed contact graph $\Gamma_{n,\alpha}$ of \mathbf{X}'_n has clique number at most $\omega(\mathcal{B}_n) + 3 = 5$ and chromatic number at least $\chi(\mathcal{B}_n) > n$.

Proof of Theorem 2. Let \mathbf{X}'_n be a 4-dimensional CAT(0) cube complex given by Proposition 14, so that the degree of 0-cubes in \mathbf{X}'_n is at most 8 and the clique number of the pointed contact graph $\Gamma_{n,\alpha}$ is at most 5. Note that $\Gamma_\#(\mathbf{X}'_n) \subseteq \Gamma_{n,\alpha} \subseteq \Gamma(\mathbf{X}'_n)$. Hence, by Proposition 13, there exists a CAT(0) cube complex \mathbf{X}_n , obtained from \mathbf{X}'_n by applying recubulation to the subgraph $\Gamma_{n,\alpha}$ of the contact graph, such that the hyperplanes of \mathbf{X}_n correspond bijectively to those of \mathbf{X}'_n , and any d -cube in \mathbf{X}_n corresponds to a d -clique in $\Gamma_{n,\alpha}$. Hence $d \leq 5$ for each d -cube, i.e. $\dim \mathbf{X}_n \leq 5$, and this bound is realized for some n . By Proposition 13(ii), $\Gamma_\#(\mathbf{X}_n) = \Gamma_{n,\alpha}$, and hence

$$\chi(\Gamma_\#(\mathbf{X}_n)) = \chi(\Gamma_{n,\alpha}) > n.$$

The cube complex \mathbf{X}_n is thus finite, and any isometric embedding of \mathbf{X}_n into the Cartesian product of trees requires at least $n + 1$ trees. Moreover, by Proposition 13(iii) the clique number of the contact graph of \mathbf{X}_n is at most $\Delta^2 + \Delta = 8^2 + 8 = 72$ for each n .

By Proposition 13, there is an isometric embedding $\mathbf{X}'_n \rightarrow \mathbf{X}_n$, and hence \mathbf{X}_n contains distinct 0-cubes a_n and b_n that are the images of α_n and β_n respectively. Let \mathbf{X} be formed from $\bigsqcup_{n \geq 0} \mathbf{X}_n$ by identifying b_n with a_{n+1} for each n . Since each 0-cube of \mathbf{X} is contained in at most 2 of the “blocks” \mathbf{X}_n , the clique number of $\Gamma(\mathbf{X})$ is at most $\Delta^2 + \Delta$. Indeed, β_n and α_{n+1} have degree at most 3, and we are taking $\Delta = 8$. On the other hand, by construction,

$$\Gamma_\#(\mathbf{X}) = \bigsqcup_{n \geq 0} \Gamma_\#(\mathbf{X}_n)$$

and hence

$$\chi(\Gamma_\#(\mathbf{X})) \geq \chi(\Gamma_\#(\mathbf{X}_n)) > n$$

for all n , i.e. the chromatic number of the crossing graph of \mathbf{X} is infinite. Thus \mathbf{X} is a 5-dimensional, uniformly locally finite CAT(0) cube complex that does not admit an isometric embedding in the Cartesian product of finitely many trees. \square

Theorem 1 states that a uniformly locally finite CAT(0) cube complex of dimension at most 2 embeds isometrically in the product of finitely many trees, while Theorem 2 asserts the existence of a 5-dimensional uniformly locally finite CAT(0) cube complex not admitting such an embedding, and it is natural to ask about the situation in dimensions 3 and 4:

Question 4. *Are 3- or 4-dimensional CAT(0) cube complexes, whose 1-skeleta have uniformly bounded degree, embeddable in the Cartesian product of finitely many trees?*

If Question 4 has a positive answer in the 3-dimensional case, then to find this answer it seems quite likely that one have to be able to colour the imprints of a collection of hyperplanes on a given hyperplane H . This hyperplane H is a 2-dimensional CAT(0) cube complex and the imprints in H are convex (gated) subcomplexes of H . Therefore, as an warm-up to Question 4, one can ask for the following common generalization of Asplund-Grünbaum’s result for axis-parallel rectangles and of the fact that any family \mathcal{T} of subtrees of a tree can be coloured with $\omega(\mathcal{T})$ colours:

Question 5. *Let \mathcal{C} be a collection of convex subcomplexes of a 2-dimensional CAT(0) cube complex. Is \mathcal{C} χ -bounded?*

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